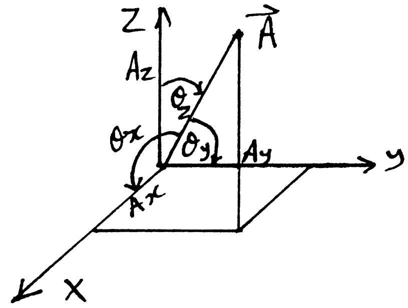
Unit 2 lesson 1

Equation of a st. line in space

Remember



$$\vec{A} = (A_x , A_y , A_z)$$

$A_x = \|\vec{A}\| \cos\theta_x$

$A_y = \|\vec{A}\| \cos\theta_y$

$A_z = \|\vec{A}\| \cos\theta_z$

$\therefore \vec{A} = \|\vec{A}\| \cos\theta_x \, \hat{i} + \|\vec{A}\| \cos\theta_y \, \hat{j} + \|\vec{A}\| \cos\theta_z \, \hat{k}$

$\therefore \vec{A} = \|\vec{A}\| (\cos\theta_x \hat{i} + \cos\theta_y \hat{j} + \cos\theta_z \hat{k})$

→ $(\theta_x , \theta_y , \theta_z)$ are called the direction angles of $\vec{A}$

→ $\cos\theta_x , \cos\theta_y , \cos\theta_z$ are called the direction cosines of $\vec{A}$

→ $\cos\theta_x \hat{i} + \cos\theta_y \hat{j} + \cos\theta_z \hat{k}$ is the unit vector in the direction of vector $\vec{A}$ $\Rightarrow \therefore \cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$

Let $l = \cos\theta_x$, $m = \cos\theta_y$, $n = \cos\theta_z$

$\therefore \rightarrow l^2 + m^2 + n^2 = 1$

$\rightarrow \vec{u} = l\hat{i} + m\hat{j} + n\hat{k}$ is the unit vector in the direction of the st. line

Note: any vector parallel to $\vec{u}$ is called the "direction Vector" of the st. line

$$\therefore d = K(l\hat{i} + m\hat{j} + n\hat{k}) = (a, b, c)$$

where a, b, c are proportional to l, m, n , $K \in R^+$

a, b, c are called direction ratio (direction numbers)

eg: If $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ are the direction cosines of a st. line then the vector $\vec{d} = K(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ is the direction vector of the st. line where $K \neq 0$

Put $K = 3 \rightarrow \vec{d} = (2, 1, 2)$

Put $K = -6 \rightarrow \vec{d} = (-4, -2, -4)$

there are infinite direction vectors each is parallel to the st. line

V.I Direction Cosines and Directionratios

Direction cosines : $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$

Direction Cosines : l , m , n

Direction ratios : a , b , c

$\therefore \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = K$ constant

$l = aK$, $m = bK$, $n = cK$

$l^2 + m^2 + n^2 = 1$

$a^2K^2 + b^2K^2 + c^2K^2 = 1$

$K^2(a^2 + b^2 + c^2) = 1$

$\therefore K = \pm \frac{1}{\sqrt{a^2+b^2+c^2}}$

$\therefore l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}$

$\therefore m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}$

$\therefore n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$

Example 1 find the direction vector of the st. line passing through $A(-2,3,1)$, $B(0,4,-2)$

Solution

A direction vector of the st. line $= \overrightarrow{AB} = \vec{B} - \vec{A}$

$= (0,4,-2) - (-2,3,1) = (2,1,-3)$

Critical thinking

① What can you say about the st. line with direction vector $\vec{d} = (a,b,0)$?

Solution

the direction vector $(a,b,0)$ is for st. line paralle to xy plane

② Find a direction vector for each of the cartesian axes.

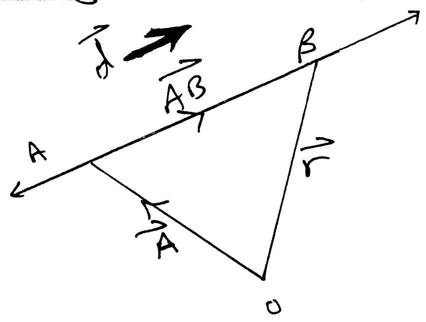
Solution

the direction vector of x-axis is $\hat{i} = (1,0,0)$

" " " " y-axis is $\hat{j} = (0,1,0)$

" " " " z-axis is $\hat{k} = (0,0,1)$

The different forms of the equation of a st. line in space



$\vec{r} = \vec{A} + \vec{AB}$ $(\vec{AB} = t\,\vec{d})$

$\vec{r} = \vec{A} + t\vec{d} \longrightarrow$ vector form

$(x, y, z) = (x_1, y_1, z_1) + t(a, b, c)$

$\therefore$
$$\left.\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}\right\} \text{parametric equations}$$

$\therefore\ t = \dfrac{x - x_1}{a}$, $t = \dfrac{y - y_1}{b}$, $t = \dfrac{z - z_1}{c}$

$\therefore\ \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} \longrightarrow$ Cartesian form

Example 1 find the different forms of the equation of the st. line passing through the point $(2, 3, -1)$ and $(-1, 2, 3)$ is a direction vector

Solution

$\vec{r} = \vec{A} + t\vec{d}$, $A = (2, 3, -1)$, $\vec{d} = (-1, 2, 3)$

$\vec{r} = (2, 3, -1) + t(-1, 2, 3) \longrightarrow$ vector form

$(x, y, z) = (2, 3, -1) + t(-1, 2, 3)$

$\therefore$
$$\left.\begin{aligned} x &= 2 - t \\ y &= 3 + 2t \\ z &= -1 + 3t \end{aligned}\right\} \text{parametric equations}$$

$\dfrac{x - 2}{-1} = \dfrac{y - 3}{2} = \dfrac{z + 1}{3} \rightarrow$ Cartesian form.

Example 2 find the different forms of the equation of the st. line passing through the two points $(2, -1, 5)$, $(-3, 1, 4)$

Solution

$d = (-3,1,4) - (2,-1,5) = (-5,2,-1)$

$\vec{r} = A + t\vec{d}$

$\vec{r} = (2,-1,5) + t(-5,2,-1) \longrightarrow$ vector form

$(x,y,z) = (2,-1,5) + t(-5,2,-1)$

$$\left.\begin{aligned} x &= 2-5t \\ y &= -1+2t \\ z &= 5-t \end{aligned}\right\} \text{parametric equations}$$

$\therefore \dfrac{x-2}{-5} = \dfrac{y+1}{2} = \dfrac{z-5}{-1} \longrightarrow$ Cartesian form.

Example (3) find the different forms of the equation of a st. line passes through the point $(-3,4,5)$ and its direction angles $(30°, 90°, 60°)$

Solution

Unit vector in the direction of the st. line

$= (\cos 30, \cos 90, \cos 60) = \left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$

$\vec{r} = A + t\vec{d}$

$\vec{r} = (-3,4,5) + t\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) \longrightarrow$ Vector form

$(x,y,z) = (-3,4,5) + t\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$

$$\left.\begin{aligned} x &= -3 + \frac{\sqrt{3}}{2}t \\ y &= 4 \\ z &= 5 + \frac{1}{2}t \end{aligned}\right\} \text{parametric equations}$$

$y = 4, \quad \dfrac{x+3}{\frac{\sqrt{3}}{2}} = \dfrac{z-5}{\frac{1}{2}}$

$\therefore y = 4, \quad \dfrac{2x+6}{\sqrt{3}} = \dfrac{2z-10}{1} \longrightarrow$ Cartesian form

Remark (1) from the previous example $\left(\overset{a}{\frac{\sqrt{3}}{2}}, \overset{b}{0}, \overset{c}{\frac{1}{2}}\right) \rightarrow b=0$

if $b=0$ then the catesion form is $\dfrac{x-x_1}{a} = \dfrac{z-z_1}{c}, \ y = y_1$

if $c=0$ then " " " " $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b}, \ z = z_1$

if $a=0$ then " " " " $\dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}, \ x = x_1$

Example 4 find the different forms of the equation of the st. line passing through the point $(3, 4, 1)$ parallel to X-axis

Solution

the unit vector in the direction of X-axis $(1, 0, 0)$

$\vec{r} = A + t\vec{d}$

$\vec{r} = (3, 4, 1) + t(1, 0, 0)$ vector form

$(x, y, z) = (3, 4, 1) + t(1, 0, 0)$

$x = 3 + t \quad , \quad y = 4 , \quad z = 1$

the cartesion equation $y = 4, z = 1$

Remark 2

1) If the st.line parallel to X-axis then the cartesian form of the equation is $y = y_1 \;,\; z = z_1$

2) If the st.line parallel to y-axis then the cartesian form of the eq. is $x = x_1 \;,\; z = z_1$

3) If the st.line parallel to z-axis then the cartesian form of the eq: is $x = x_1 \;,\; y = y_1$

Example 5 find the different forms of the equation of the st.line passing through the point $(-2, 1, 3)$ and makes equal angles with the coordinat axis

Solution

$\because \cos\theta_x = \cos\theta_y = \cos\theta_z \quad , \quad \because \cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$

$\therefore 3\cos^2\theta_x = 1 \rightarrow \cos^2\theta_x = \frac{1}{3} \quad \therefore \cos\theta_x = \pm\frac{1}{\sqrt{3}}$

$\therefore \cos\theta_x = \cos\theta_y = \cos\theta_z = \pm\frac{1}{\sqrt{3}}$

$\therefore$ the unit vector in the direction of the st. line $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\therefore (1, 1, 1)$ is vector direction of the same st. line

$\therefore$ the vector form is: $\vec{r} = (-2, 1, 3) + t(1, 1, 1)$

the parametric equations: $x = -2 + t$

$y = 1 + t$

$z = 3 + t$

the cartesian equations: $x + 2 = y - 1 = z - 3$

Remark 3 ① $\because$ a, b, c proportional to l, m, n then

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

② the equation of X-axis is $y = 0, z = 0$

the equation of y-axis is $x = 0, z = 0$

the equation of z-axis is $x = 0, y = 0$

③ If the st. line passes through the origin point then

$\vec{r} = t(a, b, c) \longrightarrow$ vector form

$x = at, \ y = bt, \ z = ct \longrightarrow$ parametric equation

$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \longrightarrow$ Cartesian form.

Example 6 find the different forms of the equation of a st. line and find a point on it

① passes through the origin point, its direction vector is $(2, -1, 7)$

② its vector form is $\vec{r} = (-1, 2, 5) + t(-1, 4, 3)$

Solution

① the vector form is $\vec{r} = t(2, -1, 7)$

the parametric equations $x = 2t, y = -t, z = 7t$

the cartesian equation: $\frac{x}{2} = \frac{y}{-1} = \frac{z}{7}$

Put $x = 2 \longrightarrow \therefore y = -1, z = 7$

$\therefore (2, -1, 7)$ is a point on the st. line

② the parametric equations:

$x = -1 - K \quad , y = 2 + 4K \, , \quad z = 5 + 3K$

the cartesian equation $\frac{x+1}{-1} = \frac{y-2}{4} = \frac{z-5}{3}$

Put $x = -1$ then $y = 2$, $z = 5$

$\therefore (-1, 2, 5)$ is a point on the st. line

Example 7 find the different forms of the equation of a st. line then find a point on it whose Cartesian eqution is $\frac{3x+1}{2} = \frac{y-1}{2} = \frac{5-z}{3}$

Solution

$$\frac{3x+1}{2} = \frac{y-1}{2} = \frac{5-z}{3} = t$$

$$\therefore \frac{3x+1}{2} = \frac{t}{1} \rightarrow 3x+1 = 2t \longrightarrow x = \frac{-1}{3} + \frac{2}{3}t$$

$$\frac{y-1}{2} = \frac{t}{1} \rightarrow y-1 = 2t \longrightarrow y = 1 + 2t$$

$$\frac{5-z}{3} = \frac{t}{1} \rightarrow 5-z = 3t \longrightarrow z = 5 - 3t$$

Parametric equation

from the parametric equations $(x, y, z) = (\frac{-1}{3}, 1, 5) + t(\frac{2}{3}, 2, -3)$

$\therefore \vec{r} = (\frac{-1}{3}, 1, 5) + t(\frac{2}{3}, 2, -3) \rightarrow$ vector form

$\boxed{\text{or}} \rightarrow (\frac{2}{3} \times 3, 2 \times 3, -3 \times 3) = (2, 6, -9)$ is dir. vector also

$\therefore \vec{r} = (\frac{-1}{3}, 1, 5) + t(2, 6, -9) \longrightarrow$ vector form

Example 8 find the different forms of the equation of the following st. line and find a point on it if its Cartesian equation is

$$y = 5 \, , \quad \frac{x-4}{3} = \frac{2z+1}{6}$$

Solution

Put $\frac{x-4}{3} = \frac{2z+1}{6} = t$

$\therefore x = 4 + 3t \, , \ y = 5 , \ z = \frac{-1}{2} + 3t \rightarrow$ parametric eqn

$\therefore \vec{r} = (4, 5, \frac{-1}{2}) + t(3, 0, 3) \rightarrow$ Vector equation

Put $x=7$ $\therefore$ $z=2\frac{1}{2}$, $y=5$

$\therefore$ $(7, 5, 2\frac{1}{2})$ are the coordinates of a point on the st.line

Example (9) find the equation of the st.line passing through the point $A(2,3,-5)$ parallel to the st.line passing through the two points $B(-1,3,2)$, $C(2,-1,3)$.

Prove that the st. line passes through the point $(17,-17,0)$

Solution

$\because$ the st.line parallel to $\overleftrightarrow{BC}$

$\therefore$ $\overrightarrow{BC}$ is the direction vector to it

$\therefore$ $\overrightarrow{BC} = C - B = (2,-1,3)-(-1,3,2) = (3,-4,1)$

$\therefore$ the vector eq: $\vec{r} = (2,3,-5)+t(3,-4,1)$

the parametric eq: $x = 2+3t$, $y = 3-4t$, $z = -5+t$

the Cartesian eq: $\frac{x-2}{3} = \frac{y-3}{-4} = \frac{z+5}{1}$

put $x = 17$, $y = -17$, $z = 0$

$\therefore$ $\frac{17-2}{3} = 5$, $\frac{-17-3}{-4} = 5$, $0+5 = 5$

$\therefore$ the st.line passes through the point $(17,-17,0)$

Example (10) find the equation of the st.line passing through the intersection point of the two lines: $L_1: \vec{r_1} = (1,-1,2)+t_1(3,-2,5)$,
$L_2: \vec{r_2} = (8,-2,14)+t_2(1,3,2)$ and the point $(2,-3,1)$

Solution

at the intersection point $\vec{r_1} = \vec{r_2}$

$\therefore$ $(1,-1,2)+t_1(3,-2,5) = (8,-2,14)+t_2(1,3,2)$

$\therefore$ $1+3t_1 = 8+t_2 \longrightarrow 3t_1 - t_2 = 7$
$-1-2t_1 = -2+3t_2 \longrightarrow -2t_1 - 3t_2 = -1$

Solve $t_1 = 2$, $t_2 = -1$

$\therefore$ the intersection point is

$$(x, y, z) = (1, -1, 2) + 2(3, -2, 5) = (7, -5, 12)$$

$\therefore$ the direction vector of the st. line is

$$(2, -3, 1) - (7, -5, 12) = (-5, 2, -11)$$

the vector equation $\vec{r} = A + t\vec{d}$

$$\vec{r} = (2, -3, 1) + t(-5, 2, -11)$$

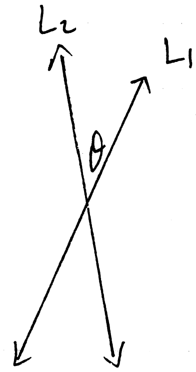
the parametric eq: $x = 2 - 5t$

$y = -3 + 2t$

$z = 1 - 11t$

the cartesian eq: $\frac{x-2}{-5} = \frac{y+3}{2} = \frac{z-1}{-11}$

The angle between 2 st. lines in space



If L_1, L_2 are two st. lines in space whose directions $\vec{d_1} = (a_1, b_1, c_1)$, $d_2 = (a_2, b_2, c_2)$, θ is the smallest angle between the two lines

$$\therefore \cos\theta = \frac{|\vec{d_1} \cdot \vec{d_2}|}{\|\vec{d_1}\| \; \|\vec{d_2}\|}, \qquad 0 \leq \theta \leq 90$$

and if (l_1, m_1, n_1), (l_2, m_2, n_2) are the direction cosines for the two st. lines then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

Example (1) find the measure of the angle between the two lines

L_1: passing through the two points $(5, 3, 5)$, $(-1, 7, 3)$

L_2: " " " " " $(2, 5, 4)$, $(-2, 3, 1)$

Solution

$$\vec{d_1} = (-1, 7, 3) - (5, 3, 5) = (-6, 4, -2)$$

$$\vec{d_2} = (-2, 3, 1) - (2, 5, 4) = (-4, -2, -3)$$

$$\cos\theta = \frac{|\vec{d_1} \cdot \vec{d_2}|}{\|\vec{d_1}\| \, \|\vec{d_2}\|}$$

$$\cos\theta = \frac{|(-6,4,-2)\cdot(-4,-2,-3)|}{\sqrt{(-6)^2+(4)^2+(-2)^2}\,\sqrt{(-4)^2+(-2)^2+(-3)^2}} = \frac{|24-8+6|}{2\sqrt{14}\times\sqrt{29}}$$

$$\cos\theta = \frac{11}{\sqrt{406}} \qquad \therefore \theta = 56^\circ\, 54'\, 45''$$

Example 2 Find the measure of the angle between the two lines:

$$\vec{r_1} = (2,-1,3) + t_1(-2,0,2), \text{ and } x=1,\ \frac{y-4}{3} = \frac{z+5}{-3}$$

Solution

from the equation of the first st. line $\vec{d_1} = (-2,0,2)$

" " " " " second st. line $\vec{d_2} = (0,3,-3)$

$$\cos\theta = \frac{|(-2,0,2)\cdot(0,3,-3)|}{\sqrt{(-2)^2+0^2+2^2}\,\sqrt{0^2+3^2+(-3)^2}} = \frac{6}{\sqrt{8}\,\sqrt{18}} = \frac{1}{2} \quad \therefore \theta = 60^\circ$$

Example 3 find the measure of the angle between the two st. lines:

$$L_1: x = 2-5t,\ y = 1-t,\ z = 3+4t,\quad L_2: \frac{x+1}{3} = \frac{2-y}{4} = \frac{z}{2}$$

Solution

$$L_1: x = 2-5t,\ y = 1-t,\ z = 3+4t, \qquad \frac{2-y}{4}\,\frac{\times -1}{\times -1} = \frac{y-2}{-4}$$

$$\vec{d_1} = (-5,-1,4), \qquad \vec{d_2} = (3,-4,2) \quad \text{complete}$$

Example 4 find the measure of the angle between the two lines whose direction cosines are

$$\Big(\underset{l_1}{\frac{5}{13\sqrt{2}}},\ \underset{m_1}{\frac{-12}{13\sqrt{2}}},\ \underset{n_1}{\frac{1}{\sqrt{2}}}\Big),\ \Big(\underset{l_2}{\frac{-3}{5\sqrt{2}}},\ \underset{m_2}{\frac{4}{5\sqrt{2}}},\ \underset{n_2}{\frac{1}{\sqrt{2}}}\Big)$$

Solution

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| = \left|\frac{5}{13\sqrt{2}}\times\frac{-3}{5\sqrt{2}} + \frac{-12}{13\sqrt{2}}\times\frac{4}{5\sqrt{2}} + \frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\right|$$

$$\cos\theta = \frac{1}{65} \Rightarrow \theta = 89^\circ\, 7'\, 6''$$

Parallel lines in space

If $\vec{d_1} = (a_1, b_1, c_1)$, $\vec{d_2} = (a_2, b_2, c_2)$ are the direction vectors of two st. lines L_1 and L_2 then $L_1 // L_2$ if

① $\vec{d_1} = k\,\vec{d_2}$ or ② $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or ③ $\vec{d_1} \times \vec{d_2} = \vec{0}$

Note ① If the two st. lines are parallel and there is a point on one of them satisfying the equation of the other, then the two st. lines are concident

② If $\vec{d_1}$ is not parallel to $\vec{d_2}$ then $\overline{L_1}$ and $\overline{L_2}$ are eighter intersect or skew

Example 5 If the two lines are parallel then find a, b

$L_1: \vec{r_1} = (2, 3, -4) + t(2, 3, a)$, $L_2: \frac{x-5}{b} = \frac{y+4}{6} = \frac{z-4}{2}$

Solution

$\vec{d_1} = (2, 3, a)$, $\vec{d_2} = (b, 6, 2)$

$\because L_1 // L_2 \quad \therefore \frac{2}{b} = \frac{3}{6} = \frac{a}{2} \longrightarrow b = 4, a = 1$

Example 6 Prove that the two st. lines intersect at a point, then find their intersection point

$\vec{r_1} = \hat{j} + t_1(\hat{i} + 2\hat{j} - \hat{k})$, $\vec{r_2} = (\hat{i} + \hat{j} + \hat{k}) + t_2(-2\hat{i} - 2\hat{j})$

Solution

$\vec{d_1} = (1, 2, -1)$, $d_2 = (-2, -2, 0)$

$\frac{a_1}{a_2} = \frac{1}{-2} = -\frac{1}{2}$, $\frac{b_1}{b_2} = \frac{2}{-2} = -1 \quad \therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\therefore$ the two st. lines are not parallel

at the intersection point $\vec{r_1} = \vec{r_2}$

$\therefore \hat{j} + t_1(\hat{i} + 2\hat{j} - \hat{k}) = (\hat{i} + \hat{j} + \hat{k}) + t_2(-2\hat{i} - 2\hat{j})$

$\therefore t_1 = 1 + 2t_2$, then $t_1 + 2t_2 = 1 \rightarrow ①$

$2t_1 = -2t_2$, then $t_1 + t_2 = 0 \rightarrow ②$

$-t_1 = 1$ then $t_1 = -1$

from ① $-1 + 2t_2 = 1 \rightarrow t_2 = 1$

this values satisfy equation ②

at the intersection point $t_1 = -1$, $t_2 = 1$

$\vec{r} = \hat{j} - 1(\hat{i} + 2\hat{j} - \hat{k})$

$\vec{r} = \hat{j} - \hat{i} - 2\hat{j} + \hat{k} = -\hat{i} - \hat{j} + \hat{k} = \underset{\text{the intersection point}}{(-1, -1, 1)}$

Perpendicular lines in space

If $\vec{d}_1 = (a_1, b_1, c_1)$, $\vec{d}_2 = (a_2, b_2, c_2)$ are the direction vectors of the two st. lines L_1, L_2 then
$L_1 \perp L_2$ if and only if $\vec{d}_1 \cdot \vec{d}_2 = 0$

Note ① Two parallel lines lie in the same plane
② two intersecting lines lie in the same plane
③ Two perpendicular lines (two cases)
1) intersecting and perpendicular then they lie in the same plane
2) not intersecting and perpendicular they they are skew perpendicular lies then they can not be put in the same plane.

Example (7) Prove that the two lines are orthogonal then show that they are skew

$\vec{r}_1 = (1, 2, 4) + t_1(2, -1, 1)$, $\vec{r}_2 = (1, 1, 1) + t_2(-2, 7, 11)$

Solution

$\vec{d}_1 = (2,-1,1)$, $\vec{d}_2 = (-2,7,11)$

$\vec{d}_1 \cdot \vec{d}_2 = (2,-1,1)\cdot(-2,7,11) = -4-7+11 = 0$

$\therefore L_1 \perp L_2$

To prove that the two st. lines are skew, we prove that there are not any values for t_1, t_2 make $\vec{r}_1 = \vec{r}_2$

Put $\vec{r}_1 = \vec{r}_2 \rightarrow (1,2,4)+t_1(2,-1,1) = (1,1,1)+t_2(-2,7,11)$

$\therefore 1+2t_1 = 1-2t_2 \longrightarrow t_1+t_2 = 0$ ①

$2-t_1 = 1+7t_2 \longrightarrow -t_1-7t_2 = -1$ ②

$4+t_1 = 1+11t_2 \longrightarrow t_1 - 11t_2 = -3$ ③

by solving ①, ② $t_1 = \frac{-1}{6}$, $t_2 = \frac{1}{6}$ and these values do not satisfy the third equation

$\therefore$ the two lines are skew (perpendicular and skew)

Example 8 Prove that the two st. lines are skew

$\vec{r}_1 = (3,-1,2)+t_1(4,1,3)$, $\vec{r}_2 = (0,4,-1)+t_2(1,-1,2)$

Solution

Idea : To prove that the two st. lines are skew

① Prove that they are not parallel

② " " " " " intersecting

$\vec{d}_1 = (4,1,3)$, $\vec{d}_2 = (1,-1,2)$

$\frac{a_1}{a_2} = \frac{4}{1} = 4$, $\frac{b_1}{b_2} = \frac{1}{-1} = -1$ $\quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\therefore L_1$ not parallel to line L_2

To find the intersection point put $\vec{r}_1 = \vec{r}_2$

$(3,-1,2)+t_1(4,1,3) = (0,4,-1)+t_2(1,-1,2)$

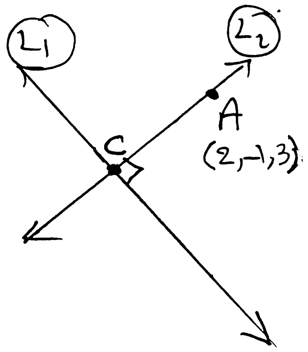
$\therefore 3+4t_1 = 0+t_2 \longrightarrow 4t_1 - t_2 = -3 \longrightarrow$ ①

$-1+t_1 = 4-t_2 \longrightarrow t_1+t_2 = 5 \longrightarrow$ ②

$2+3t_1 = -1+2t_2 \longrightarrow 3t_1 - 2t_2 = -3 \longrightarrow$ ③

by solving ①, ② $t_1 = \frac{2}{5}$, $t_2 = \frac{23}{5}$ and these values do not satisfy the third equation $\therefore$ the two lines are skew

Example 9 find the equation of the st. line passing through the point $(2,-1,3)$ and intersects the st. line $\vec{r}_1 = (1,-1,2) + t(2,2,-1)$ or thogonally



Solution

let c be the intersection point

$\therefore C \in L_1$ (given st. line)

$\therefore$ c can be written in the form

$$C = (1+2t, -1+2t, 2-t)$$

$\vec{d}_2$ is the direction vector of L_2 (the required st. line)

$$\vec{d}_2 = \vec{AC} = \vec{c} - \vec{A} = (2t-1, 2t, -t-1)$$

$\because \vec{d}_1 = (2,2,-1)$

$\because$ the two st. lines are perpendicular $\therefore \vec{d}_1 \cdot \vec{d}_2 = 0$

$$\therefore (2,2,-1) \cdot (2t-1, 2t, -t-1) = 0$$

$$2(2t-1) + 2(2t) - 1(-t-1) = 0$$

$$4t - 2 + 4t + t + 1 = 0 \rightarrow 9t = 1 \rightarrow t = \frac{1}{9}$$

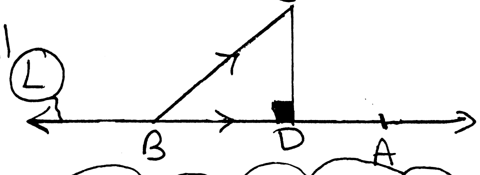
$$\therefore \vec{d}_2 = \left(\frac{-7}{9}, \frac{2}{9}, \frac{-10}{9}\right)$$

or we can write $\vec{d}_2 = \left(\frac{-7}{9} \times 9, \frac{2}{9} \times 9, \frac{-10}{9} \times 9\right) = (-7, 2, -10)$

$\therefore$ the equation of L_2 is $\vec{r} = (2,-1,3) + t_2(-7, 2, -10)$

The distance between a point and a st.line in space

To find CD



① find $\vec{BC} = \vec{C} - \vec{B}$ then find $BC = \|\vec{BC}\|$

② find BD = the absolue value of the projection of $\vec{BC}$ on $\vec{BA}$

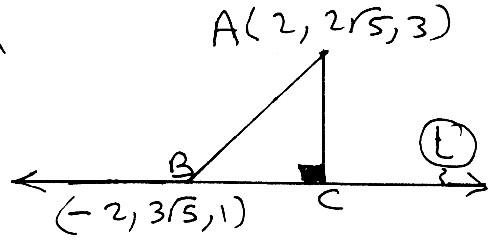
$$BD = \frac{|\vec{BC} \cdot \vec{BA}|}{\|\vec{BA}\|}$$

مباشرة $\boxed{\text{or}}$ $CD = \dfrac{\|\vec{BC} \times \vec{d}\|}{\|\vec{d}\|}$

③ find $CD = \sqrt{(BC)^2 - (BD)^2}$ "Phythagorus' theorem"

Example (1) find the distance between the point $A(2, 2\sqrt{5}, 3)$ and the st. line L whose equation is
$\vec{r} = (-2, 3\sqrt{5}, 1) + t(8, -\sqrt{5}, 2)$

Solution

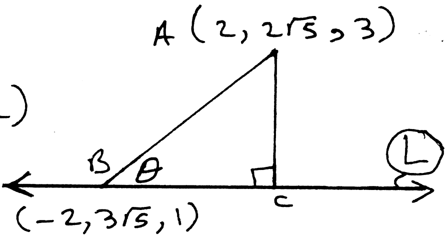


$\overrightarrow{BA} = \vec{A} - \vec{B} = (2, 2\sqrt{5}, 3) - (-2, 3\sqrt{5}, 1)$
$\overrightarrow{BA} = (4, -\sqrt{5}, 2)$
$BA = \|\overrightarrow{BA}\| = \sqrt{4^2 + (-\sqrt{5})^2 + 2^2} = 5$
BC = the length of the projection of $\overrightarrow{BA}$ on $\overleftrightarrow{L}$

$$\therefore BC = \frac{|\overrightarrow{BA} \cdot \vec{d}|}{\|\vec{d}\|} = \frac{|(4, -\sqrt{5}, 2) \cdot (8, -\sqrt{5}, 2)|}{\sqrt{8^2 + (-\sqrt{5})^2 + 2^2}} = \frac{|32 + 5 + 4|}{\sqrt{64 + 5 + 4}}$$

$$BC = \frac{41}{\sqrt{73}} \qquad \therefore AC = \sqrt{5^2 - \left(\frac{41}{\sqrt{73}}\right)^2} = \frac{12}{\sqrt{73}} \simeq 1.4 \text{ length unit}$$

<u>another solution</u>



$\vec{d_1}$ is the directed vector of $\overleftrightarrow{L} = (8, -\sqrt{5}, 2)$
$\vec{d_2}$ is the directed vector of $\overrightarrow{BA} = (4, -\sqrt{5}, 2)$

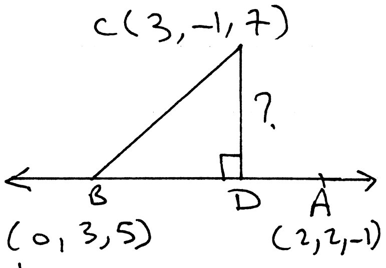
$$\cos\theta = \frac{|(8, -\sqrt{5}, 2) \cdot (4, -\sqrt{5}, 2)|}{\sqrt{64 + 5 + 4}\ \sqrt{16 + 5 + 4}} \qquad \therefore \theta = 16^\circ 19'$$

$\because AB = 5$ unit length
$AC = AB \sin\theta = 5 \sin 16^\circ 19' \simeq 1.4$

Note the distance between point $A(x_1, y_1, z_1)$ and
- x-axis is $\sqrt{y_1^2 + z_1^2}$
- y-axis is $\sqrt{x_1^2 + z_1^2}$
- z-axis is $\sqrt{x_1^2 + y_1^2}$

Example 2 find the perpendicular distance from point $(3,-1,7)$ to the st. line passing through the two points $(2,2,-1)$ and $(0,3,5)$

Solution



$\overrightarrow{BC} = \vec{C} - \vec{B} = (3,-1,7) - (0,3,5) = (3,-4,2)$

$\vec{d} = \overrightarrow{BA} = \vec{A} - \vec{B} = (2,-1,-6)$

BD is the length of the projection of $\overrightarrow{BC}$ on $\overrightarrow{BA}$ $(0,3,5)$

$$BD = \frac{|\overrightarrow{BC} \cdot \overrightarrow{BA}|}{\|\overrightarrow{BA}\|} = \frac{|(3,-4,2)\cdot(2,-1,-6)|}{\sqrt{(2)^2+(-1)^2+(-6)^2}} = \frac{2}{\sqrt{41}}$$

$$\|\overrightarrow{BC}\| = \sqrt{(3)^2+(-4)^2+2^2} = \sqrt{29}$$

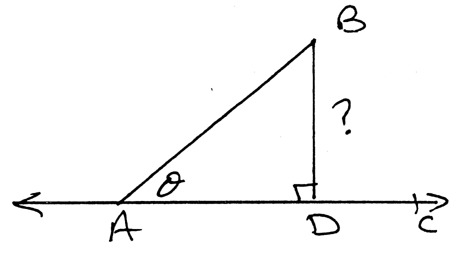
$$CD = \sqrt{(\sqrt{29})^2 - \left(\frac{2}{\sqrt{41}}\right)^2} \simeq 5.3 \text{ unit length}$$

Critical thinking SB. P. 158

Can you prove the following relation which identifies the distance from point B to the st. line $\vec{r} = \vec{A} + t\vec{d}$

$$\text{the perpendicular distance} = \frac{\|\overrightarrow{AB} \times \vec{d}\|}{\|\vec{d}\|}$$

Solution



$$\sin\theta = \frac{\|\overrightarrow{BD}\|}{\|\overrightarrow{AB}\|} \longrightarrow \|\overrightarrow{BD}\| = \|\overrightarrow{AB}\| \sin\theta$$

$$BD = \|\overrightarrow{AB}\| \sin\theta \times \frac{\|\vec{d}\|}{\|\vec{d}\|}$$

$$BD = \frac{\|\overrightarrow{AB}\| \, \|\vec{d}\| \sin\theta}{\|\vec{d}\|} = \frac{\|\overrightarrow{AB} \times \vec{d}\|}{\|\vec{d}\|}$$

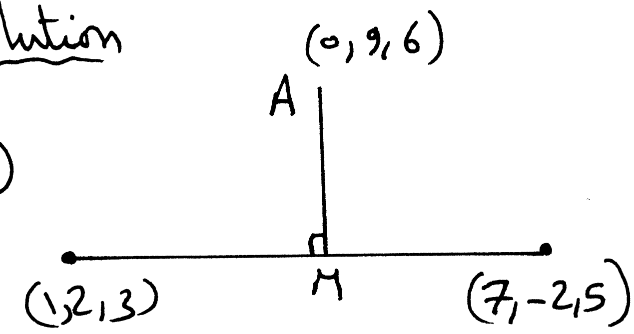
from Example 2 $\overrightarrow{BC} = (3,-4,2)$, $\vec{d} = (2,-1,-6)$

$$\overrightarrow{BC} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 2 \\ 2 & -1 & -6 \end{vmatrix} = 26\hat{i} + 22\hat{j} + 5\hat{k}$$

$$L \text{ of } \perp = \frac{\|\overrightarrow{BC} \times \vec{d}\|}{\|\vec{d}\|} = \frac{\sqrt{26^2+22^2+5^2}}{\sqrt{2^2+(-1)^2+(-6)^2}} \simeq 5.3 \text{ unit length}$$

find the projection of the the point $(0, 9, 6)$ to the st. line passing through the two points $(1, 2, 3)$, $(7, -2, 5)$

Solution



the given line $\vec{r_1}$

$\vec{r_1} = (1, 2, 3) + t_1 (-6, 4, -2)$

$\because M \in \vec{r_1}$

$\therefore M = (1 - 6K, 2 + 4K, 3 - 2K)$

condition of $\perp$, $\vec{MA} = \vec{d_2} = (-6K + 1, 4K - 7, -2K - 3)$

$\vec{d_1} \cdot \vec{d_2} = 0$

$(-6, 4, -2) \cdot (-6K + 1, 4K - 7, -2K - 3)$

$36K - 6 + 16K - 28 + 4K + 6 = 0$

$56K = 28 \rightarrow K = \frac{1}{2}$

$M = (1 - 6 \times \frac{1}{2}, 2 + 4 \times \frac{1}{2}, 3 - 2 \times \frac{1}{2})$

$M = (-2, 4, 2)$

N.B. 1) direction of X-axis is $\hat{i} = (1, 0, 0)$

2) direction of xy plane is $(a, b, 0)$

3) $x = 3$, $y = 8 \rightarrow$ st. line // z-axis

4) $\frac{x - 5}{2} = \frac{y - 7}{2}$, $z = 4 \rightarrow$ st. line // xy plane

* equation of st. line // X-axis $\rightarrow y = \text{const}, z = \text{const}$

* equation of st. line // xy plane $\rightarrow z = \text{constant}$

eg: the equation of st. line passes through $(2, 3, 5)$

// X-axis $\rightarrow y = 3, z = 5$

// Xy plane $\rightarrow \frac{x - 2}{a} = \frac{y - 3}{b}, z = 5$

Ideas of equation of st. line

1. $* \; x = 0 , \; y = z \rightarrow \vec{d} = (0, 1, 1)$

because

عزلناها $x = 0 , \; \frac{y}{1} = \frac{z}{1}$

$y = 0 , \; x = z \rightarrow \vec{d} = (1, 0, 1)$

2. $\underset{\text{const}}{x = 1} , \; \underset{\text{const}}{y = 5} \rightarrow \vec{d} = (0, 0, c)$ أي رقم because z is variable

parametric eq. عزلناهم

st. line // z-axis

$x = 2 , \; z = -1 \rightarrow \vec{d} = (0, b, 0) \rightarrow$ // y-axis

3. $3x + 2y = 7 , \; z = 3 \rightarrow \vec{d} = (2, -3, 0)$

عزلناها

$\text{slope} = -\frac{\text{coeff of } x}{\text{coeff of } y}$

$\text{slope} = \frac{-3}{2} = \frac{b}{a}$

4. $2x = 3y = -z \rightarrow \div -6$

$\frac{2x}{-6} = \frac{3y}{-6} = \frac{-z}{-6}$

$\frac{x}{-3} = \frac{y}{-2} = \frac{z}{6} \rightarrow \vec{d} = (-3, -2, 6)$

5. $\cos \theta_z = \frac{|A_z|}{\|\vec{A}\|}$

6. $A(1, 2, 3)$ —— $C(x, y, 7)$ —— $B(2, 10, 1)$ find c

$\vec{d}_1 = \overrightarrow{AB} = (1, 8, -2) , \; \vec{d}_2 = \overrightarrow{AC} = (x-1, y-2, 4)$

$\vec{d}_1 \parallel \vec{d}_2 \Rightarrow \frac{x-1}{1} = \frac{y-2}{8} = \frac{4}{-2} \rightarrow x = -1$

$y = -14$