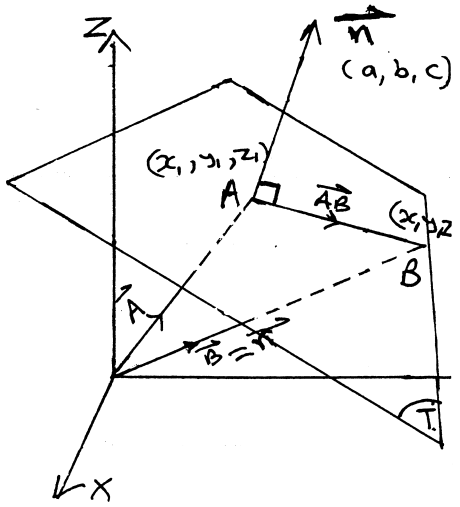
The equation of a plane in space

The vector form of the equation of a plane in space



$\vec{A}$ is the position vector of point A
$\vec{n}$ is the perpendicular direction vector to the plane

$\because \vec{n} \perp$ the plane containing $\overrightarrow{AB}$

$\therefore \vec{n} \perp \overrightarrow{AB}$

$\therefore \vec{n} \cdot \overrightarrow{AB} = 0$

$\vec{n} \cdot (\vec{B} - \vec{A}) = 0$

$\vec{n} \cdot (\vec{r} - \vec{A}) = 0 \rightarrow \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{A} \rightarrow$ (vector form of the equation of plane)

$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \rightarrow$ (standard form of the eq

$ax + by + cz - ax_1 - by_1 - cz_1 = 0$

let $-ax_1 - by_1 - cz_1 = d$

$\therefore ax + by + cz + d = 0 \rightarrow$ general form of the eq. of the plane ← given (point, $\perp$ vector)

Example [1] find the diff. forms of the equation

Type① passing through the point (2, -3, 5) and $\vec{n} = (-1, 2, 3)$ perpendicular to it

Solution

vector form $\because \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{A}$

normal vector = perpendicular vector

$(-1, 2, 3) \cdot \vec{r} = (-1, 2, 3) \cdot (2, -3, 5)$

$\therefore (-1, 2, 3) \cdot \vec{r} = 7$

standard form $\because a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$-1(x - 2) + 2(y + 3) + 3(z - 5) = 0$

general form $-x + 2y + 3z - 7 = 0$

Example 2 find the different forms of the equation of the plane passing through points $(3,-1,0)$, $(2,1,4)$, $(0,3,3)$

Type ②

Solution

First : we must make sure that the points are non-collinear

let $A(3,-1,0)$, $B(2,1,4)$, $C(0,3,3)$

$\overrightarrow{AB} = \vec{B} - \vec{A} = (-1,2,4)$, $\overrightarrow{AC} = \vec{C} - \vec{A} = (-3,4,3)$

$\because \frac{-1}{-3} \neq \frac{1}{2}$ $\therefore \overrightarrow{AB} \not\parallel \overrightarrow{AC}$ $\therefore$ the points are non-collinear

Second: To find the equation of the plane we need:

① given point (we have 3 points)

② perpendicular (normal) vector $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$\therefore \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & -1 \\ 3 & 1 & -5 \end{vmatrix} = 6\hat{i} - 13\hat{j} + \hat{k} = (6,-13,1)$$

<u>vector form:</u> $\rightarrow \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{A}$

$$(6,-13,1) \cdot \vec{r} = (6,-13,1) \cdot (1,3,5)$$

$$(6,-13,1) \cdot \vec{r} = -28$$

<u>standard form:</u> $6(x-1) - 13(y-3) + (z-5) = 0$

<u>general form:</u> $6x - 13y + z + 28 = 0$

Another solution the equation of a plane passing through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)

is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 & z_2 - z_1 \end{vmatrix} = \begin{vmatrix} x-1 & y-3 & z-5 \\ -1-1 & 2-3 & 4-5 \\ 4-1 & 4-3 & 0-5 \end{vmatrix}$$

$$= \begin{vmatrix} x-1 & y-3 & z-5 \\ -2 & -1 & -1 \\ 3 & 1 & -5 \end{vmatrix} = 0 \Rightarrow$$

standard form: $6(x-1) - 13(y-3) + (z-5) = 0$

general form: $6x - 13y + z + 28 = 0$

Vector form: $(6,-13,1) \cdot \vec{r} = -28$

Example 3 Prove that the two st. lines intersecting, then find the equation of the plane containing them

Type ③

$\vec{r_1} = (3\hat{i} + \hat{j} - \hat{k}) + t_1(\hat{i} + 2\hat{j} + 3\hat{k})$

$\vec{r_2} = (2\hat{i} + 5\hat{j}) + t_2(\hat{i} - \hat{j} + \hat{k})$

Solution

If the two st. lines intersect then $r_1 = r_2$

$\therefore (3\hat{i} + \hat{j} - \hat{k}) + t_1(\hat{i} + 2\hat{j} + 3\hat{k}) = (2\hat{i} + 5\hat{j}) + t_2(\hat{i} - \hat{j} + \hat{k})$

$\therefore 3 + t_1 = 2 + t_2$, then $t_1 - t_2 = -1 \rightarrow ①$

$1 + 2t_1 = 5 - t_2$, then $2t_1 + t_2 = 4 \rightarrow ②$ } solve

$-1 + 3t_1 = t_2$, then $3t_1 - t_2 = 1 \rightarrow ③$

by solving ①, ② $t_1 = 1$, $t_2 = 2$

by subistituting in ③, we find it satisfies the equation

$\therefore$ The two st. lines intersect

$\therefore$ the normal vector $\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{i} + 2\hat{j} - 3\hat{k}$

the vector equation $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{A}$

$\therefore (5, 2, -3) \cdot \vec{r} = (5, 2, -3) \cdot (3, 1, -1)$

$(5, 2, -3) \cdot \vec{r} = 20$

the general form $(5, 2, -3) \cdot (x, y, z) = 20$

$\therefore 5x + 2y - 3z = 20$

Example 4 Prove that the two st. lines are intersecting, then find the equation of the plane containing the two st. lines

$L_1: 2x = 3y = 4z$, $L_2: 3x = 2y = 5z$

Solution

$\boxed{2 \times 3 \times 4 = 24}$

$\because \frac{2x}{24} = \frac{3y}{24} = \frac{4z}{24} \rightarrow \frac{x}{12} = \frac{y}{8} = \frac{z}{6} \rightarrow$ st. line passing $(0, 0, 0)$

$\vec{d} = (12, 8, 6)$

$\vec{d_1} = (6, 4, 3)$

$\frac{3x}{30} = \frac{2y}{30} = \frac{5z}{30} \rightarrow \frac{x}{10} = \frac{y}{15} = \frac{z}{6}$ ∴ the st. line passing through point $(0,0,0)$ and $d_2 = (10, 15, 6)$

∴ the two st. lines intersect at the same point $(0,0,0)$

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 4 & 3 \\ 10 & 15 & 6 \end{vmatrix} = -21\hat{i} - 6\hat{j} + 50\hat{k} = (-21, -6, 50)$$

∴ the vector equation: $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{A}$

$$(-21, -6, 50) \cdot \vec{r} = (14, -11, -5) \cdot (0,0,0)$$

$$(-21, -6, 50) \cdot \vec{r} = 0$$

general form $(-20, -6, 50) \cdot (x, y, z) = 0$

$$-20x - 6y + 50z = 0$$

Example 5 find the point of intersection of the st. line $2x = 3y - 1 = z - 4$ with the plane $3x + y - 2z = 5$

Solution

from the equation of the plane : $\rightarrow y = 5 + 2z - 3x$

by subistitution in the st. line equation:

$$2x = 14 + 6z - 9x = z - 4$$

$2x = 14 + 6z - 9x$

$11x - 6z = 14 \rightarrow ①$

$14 + 6z - 9x = z - 4$

$-9x + 5z = -18 \rightarrow ②$

from ①, ② $\left.\begin{matrix} 11x - 6z = 14 \\ -9x + 5z = -18 \end{matrix}\right\}$ solve $\rightarrow x = -38$, $z = -72$

by subistitution in the plane equation $3x + y - 2z = 5$

$$3(-38) + y - 2(-72) = 5 \rightarrow y = -25$$

∴ the point of intersection is $(-38, -25, -72)$

V.I Remarks — the relation between st. line and plane in the space

when we solve the two equations of the st. line and the plane together

① If the s.s $= \phi$ then the st. line is parallel to the plane

② the solution is one point the the st. line intersects the plane at this point

③ If there are two common points between the st. line and the plane then the plane contains the st. line

Example 6

find the intersection point of the st. line $\vec{r} = (1, 4, 2) + t\,(3, 2, 2)$ with the plane $(3, 2, 2) \cdot \vec{r} = -2$

Solution

$\because$ the st. line intersects the plane then $\vec{r_1} = \vec{r_2}$

$\because (3, 2, 2) \cdot \vec{r} = -2$, $\vec{r} = (1, 4, 2) + t\,(3, 2, 2)$

$\therefore (3, 2, 2) \cdot [(1, 4, 2) + t\,(3, 2, 2)] = -2$

$\therefore (3, 2, 2) \cdot (1, 4, 2) + t\,(3, 2, 2) \cdot (3, 2, 2) = -2$

$\therefore 3 + 8 + 4 + t\,(9 + 4 + 4) = -2 \qquad \therefore t = -1$

by subistition in the equation of the st. line

$\vec{r} = (1, 4, 2) + t\,(3, 2, 2)$

$(x, y, z) = (1, 4, 2) - 1\,(3, 2, 2) = (-2, 2, 0)$

$\therefore$ the intersection point is $(-2, 2, 0)$

Equation of a plane using the intercepted parts from the coordinate axes

If a plane cuts the coordinates axes at points $(x_1, 0, 0)$, $(0, y_1, 0)$, $(0, 0, z_1)$ then the equation of the plane is in the form $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$

Proof

the equation of the plane is $ax + by + cz + d = 0$, $d \neq 0$

the points $(x_1, 0, 0)$, $(0, y_1, 0)$, $(0, 0, z_1)$ satisfy the equation

$\therefore$ By sub with $(x_1, 0, 0)$ in the eq:

Similarly " " $(0, y_1, 0)$ " " $\rightarrow b = \frac{-d}{y_1}$

" " $(0, 0, z_1)$ " " $\rightarrow c = \frac{-d}{z_1}$

by subistituting in the equation of the plane $ax + by + cz + d = 0$

we get $\frac{-d}{x_1}x - \frac{d}{y_1}y - \frac{d}{z_1}z + d = 0 \rightarrow$ take $(-d)$ common factor

$$\therefore -d\left(\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} - 1\right) = 0, \quad d \neq 0$$

$d \neq 0$ doesn't pass through the origin

$$\therefore \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} - 1 = 0$$

$$\therefore \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

eg: the equation of the st. line which intercepts the coordinate axes x, y, z the parts $2, -3, 5$ respectively is $\frac{x}{2} + \frac{y}{-3} + \frac{z}{5} = 1$

* the equation of the st. line passing through the points $(0, 0, -4)$, $(5, 0, 0)$, $(0, -2, 0)$ is

$$\frac{x}{5} + \frac{y}{-2} + \frac{z}{-4} = 1$$

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Critical thinking V.I

[1] If the plane $3x+2y+4Z=12$ intersects the coordinate axes X, y, Z at the points A, B, C respictively, find the area of Δ ABC

Solution

$3x+2y+4Z=12 \longrightarrow (\div 12)$

$$\frac{3x}{12}+\frac{2y}{12}+\frac{4Z}{12}=1$$

$$\therefore \frac{x}{4}+\frac{y}{6}+\frac{Z}{3}=1$$

$\therefore$ the plane cuts the Coordinate axes x, y, z at the points

$A(4,0,0)$, $B(0,6,0)$, $C(0,0,3)$

$\therefore \overrightarrow{AB}=\vec{B}-\vec{A}=(0,6,0)-(4,0,0)=(-4,6,0)$

$\therefore \overrightarrow{AC}=\vec{C}-\vec{A}=(0,0,3)-(4,0,0)=(-4,0,3)$

$\therefore$ area of $\Delta ABC=\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\|$

$$\therefore \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -4 & 6 & 0\\ -4 & 0 & 3\end{vmatrix}=18\hat{i}+12\hat{j}+24\hat{k}=(18,12,24)$$

$$\|\overrightarrow{AB}\times\overrightarrow{AC}\|=\sqrt{(18)^2+(12)^2+(24)^2}=6\sqrt{29}$$

$\therefore$ area of Δ ABC $=\frac{1}{2}\times 6\sqrt{29}=3\sqrt{29}$ squared unit.

Critical thinking [2] If a plane intersects the coordinate axes at the points A, B, C and the point (p, q, r) is the centroid of ΔABC (point of intersection of the medians of ΔABC). Prove that the equation of the plane is

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$$

Proof

the centroid of the triangle $= \frac{A+B+C}{3} = (p, q, r)$

$\because A = (x_1, 0, 0)$, $B = (0, y_1, 0)$, $C = (0, 0, z_1)$

$\therefore (p, q, r) = \frac{(x_1, 0, 0) + (0, y_1, 0) + (0, 0, z_1)}{3} = \frac{(x_1, y_1, z_1)}{3}$

$\therefore p = \frac{x_1}{3}$, $q = \frac{y_1}{3}$, $r = \frac{z_1}{3}$

$\therefore x_1 = 3p$, $y_1 = 3q$, $z_1 = 3r$

$\because$ the equation of the plane with the coordinate axes intercepts

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

$\therefore$ the equation is $\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1 \longrightarrow \times 3$

$\therefore$ the equation is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$

Example 2 find the general equation of the plane

$$(x, y, z) = (3, 2, 5) + t_1(-3, -3, 4) + t_2(1, 2, -1)$$

such that t_1, t_2 parametric equations

Solution

$\because (x, y, z) = (3, 2, 5) + t_1(-3, -3, 4) + t_2(1, 2, -1)$

$\therefore x = 3 - 3t_1 + t_2 \longrightarrow (1)$

$y = 2 - 3t_1 + 2t_2 \longrightarrow (2)$

$z = 5 + 4t_1 - t_2 \longrightarrow (3)$

by adding (1), (3) $\therefore x + z = 8 + t_1 \rightarrow t_1 = x + z - 8 \rightarrow (4)$

by subtracting (1) from (2) $\therefore y - x = -1 + t_2 \rightarrow t_2 = y - x + 1 \rightarrow (5)$

by subtituting from (4), (5) in (1)

$\therefore x = 3 - 3(x + z - 8) + (y - x + 1)$

$\therefore$ the general equation is $5x - y + 3z - 28 = 0$

Position of a point with respect to a plane

Example 11 find the position of each of the following points with respect to the plane $2x - 3y + 4z - 5 =$

$A(-3, 3, 5)$, $B(6, 5, 1)$, $C(6, -5, 1)$

Solution

by subistituting $A(-3, 3, 5) \rightarrow 2(-3) - 3(3) + 4(5) - 5 = 0 \therefore A \in$ the plane

$B(6, 5, 1) \rightarrow 2(6) - 3(5) + 4(1) - 5 = -4 \therefore A \notin$ the plane

$C(6, -5, 1) \rightarrow 2(6) - 3(-5) + 4(1) - 5 = 26 \therefore A \notin$ the plane

$\therefore B, C \notin$ the plane and in different sides of the plane.

V.I Remarks the general form of the equation is $ax + by + cz + d = 0$ then:

1 If $d = 0$ then the plane passes through the origin

eg. $2x + 3y + 2z = 0$

2 If $a = 0$ then the plane // x-axis and $\perp$ the plane yz

eg: $5y - 2z + 7 = 0$

3 If $d = 0$, $a = 0$ then the plane contains X-axis, $\perp$ plane yz

$5y - 2z = 0$

Eg:

① $x = 5$ is a plane parallel to yz plane

$y = 2$ " " " " xz plane

$z = 3$ " " " " xy plane

② $x = 0$ is the equation of yz plane

$y = 0$ is the equation of xz plane

$z = 0$ is the equation of xy plane

③ $2x + 3y + 4z + 7 = 0 \rightarrow$ plane does not pass throng the origin

$2x + 3y + 4z = 0 \rightarrow$ plane passing through the origin

④ $2x + 3y + 7 = 0 \rightarrow$ plane not pass $(0,0)$, // Z-axis $\perp$ Xy plane

$2x + 3y = 0 \rightarrow$ plane passes $(0,0)$ Contains Z-axis, $\perp$ Xy plane

the angle between a st. line and a given plane

If (a_1, b_1, c_1) is the direction vector of a st. line, (a_2, b_2, c_2) is the perpendicular direction vector of the plane, then the measure of the smaller angle between the st. line and the plane is the complement angle of angle θ such that

$$\cos\theta = \frac{|(a_1, b_1, c_1) \cdot (a_2, b_2, c_2)|}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

Example find the measure of the smaller angle between the st. line $\frac{x-5}{2} = \frac{y}{-1} = \frac{z+4}{3}$ and the plane $3x+2y+z-5=0$

Solution

the required angle is the complementary angle of θ.

$$\cos\theta = \frac{|(2,-1,3) \cdot (3,2,1)|}{\sqrt{4+1+9}\ \sqrt{9+4+1}} = \frac{7}{14} = \frac{1}{2}$$

$\therefore \theta^\circ = 60$ $\therefore$ the required angle $= 90 - 60 = \boxed{30^\circ}$

The relative positions of two planes in the space

If $T_1 : a_1x + b_1y + c_1z + d_1 = 0$, $T_2 : a_2x + b_2y + c_2z + d_2 = 0$ are the equations of two distinct planes in the space then $\vec{n_1} = (a_1, b_1, c_1)$ is the perpendicular directed vector to plane T_1, $\vec{n_2} = (a_2, b_2, c_2)$ is the perpendicular directed vector to plane T_2,

if the two planes T_1, T_2 are intersecting then $T_1 \cap T_2 = L$ (st. line) and then we can find the angle between the two planes which is the angle between the two perpendicular vectors $\vec{n_1}, \vec{n_2}$.

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{\|\vec{n_1}\| \, \|\vec{n_2}\|}, \quad 0 \leq \theta \leq 90$$

note that if $\vec{n_1} \cdot \vec{n_2} = 0$ then the two planes are $\perp$

The condition of perpendicular for two planes

① $\vec{n_1} \cdot \vec{n_2} = 0$ or ② $a_1a_2 + b_1b_2 + c_1c_2 = 0$

The condition of parallel planes

① $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or ② $\vec{n_1} = K\vec{n_2}$ or ③ $\vec{n_1} \times \vec{n_2} = \vec{0}$

V.I Remark

① If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$ then the two planes are congruent

② If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \neq \frac{d_1}{d_2}$ " " " " are parallel not congruent.

Example 1 find the angle between the two planes
$x+2y-3z=20$, $3x-2y+2z=30$

Solution

$\vec{n}_1=(1,2,-3)$, $\vec{n}_2=(3,-2,2)$

$$\cos\theta=\frac{|(1,2,-3)\cdot(3,-2,2)|}{\sqrt{1+4+9}\ \sqrt{9+4+4}}=\frac{\sqrt{238}}{34}\quad \therefore \theta\simeq 63^\circ 1'$$

Example 2 find K which makes the planes $\perp$
$3x-4y+2z+9=0$, $3x+4y-Kz+7=0$

Solution

$\vec{n}_1=(3,-4,2)$, $\vec{n}_2=(3,4,-K)$

$\vec{n}_1\cdot\vec{n}_2=0 \rightarrow (3,-4,2)\cdot(3,4,-K)=0$

$9-16-2K=0 \rightarrow K=-3$

Example 3 If the angle between the two planes $3x-6y+mz-4=0$, $x+z-7=0$ is 45°. find m

Solution

$\vec{n}_1=(3,-6,m)$, $\vec{n}_2=(1,0,1)$

$$\cos\theta=\frac{|(3,-6,m)\cdot(1,0,1)|}{\sqrt{9+36+m^2}\ \sqrt{1+0+1}}$$

$$\frac{1}{\cancel{\sqrt{2}}}=\frac{|3+m|}{\sqrt{45+m^2}\ \cancel{\sqrt{2}}}$$

$\therefore \sqrt{45+m^2}=|3+m|$ by squaring

$45+m^2=9+6m+m^2 \rightarrow m=6$

The equation of the intersection line between 2 planes

If $\pi_1 : a_1 x + b_1 y + c_1 z + d_1 = 0$, $\pi_2 : a_2 x + b_2 y + c_2 z + d_2 = 0$ are the equations of two distinct planes in space if $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, $\frac{c_1}{c_2}$ not all equal then the two planes are intersecting. then we can find the equation of the line between them by solving the equations together

Example (4) find the equation of the line of intersection of the two planes

$x + y + 2z + 1 = 0$, $2x + y - z + 1 = 0$

Solution

$\frac{a_1}{a_2} = \frac{1}{2}$, $\frac{b_1}{b_2} = \frac{1}{1} = 1$, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\therefore$ the two planes are intersecting

$x + y + 2z + 1 = 0$ (1)

$2x + y - z + 1 = 0$ (2)

(1) $-$ (2) $\rightarrow x - 3z = 0 \rightarrow x = 3z$ (3)

(2) $\times 2$ + (1) $\rightarrow 5x + 3y + 3 = 0 \rightarrow x = \frac{-3-3y}{5}$ (4)

from (3), (4) $x = \frac{-3-3y}{5} = 3z$ Cartesian form

another solution the line of the intersection is perpendicular to $\vec{n_1}$, $\vec{n_2}$

$\because \vec{n_1} = (1, 1, 2)$, $\vec{n_2} = (2, 1, -1)$ $\therefore$

$\therefore \vec{d} = \vec{n_1} \times \vec{n_2}$ is the direction vector of the line of the intersection $\rightarrow$

$$\therefore \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -3\hat{i} + 5\hat{j} - \hat{k} = (-3, 5, -1)$$

To find a point on the line of the intersection we put $x=2$ (or any other number) to the two equations

$\therefore y+2z=-3$ ①
$y-z=-5$ ② $\Big\}$ Solve

$\therefore z=\frac{2}{3}$, $y=\frac{-13}{3}$

$\therefore$ the point $(2, \frac{2}{3}, \frac{-13}{3})$ lies on the line of the intersection

$\therefore$ the vector form: $\vec{r}=(2, \frac{2}{3}, \frac{-13}{3})+t(-3, 5, -1)$

Third solution $x+y+2z+1=0 \rightarrow$ ①
$2x+y-z+1=0 \rightarrow$ ② $\Big\}\ominus \Rightarrow x-3z=0$
$\therefore x=3z$

let $z=k \rightarrow x=3k$

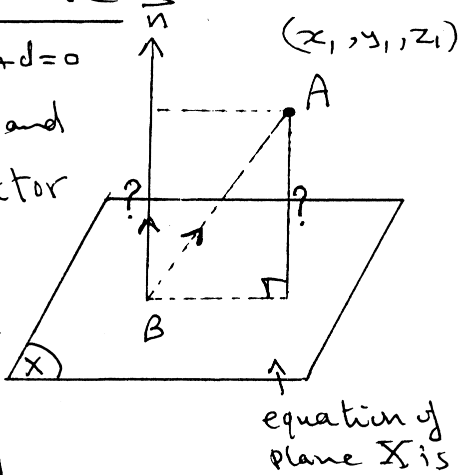
from ② $2x+y-z+1=0$

$2\times 3k+y-k+1=0 \rightarrow y=-1-5k$

$\therefore$ the parametric equations of the line of intersection are

$x=3k$, $y=-1-5k$, $z=k$

The length of the perpendicular drawn from a point to a plane



$ax+by+cz+d=0$

If $A=(x_1, y_1, z_1)$ is a point outside plan X and B is a point on the plane, $\vec{n}$ is the normal vector to the plane, then the distance from the point A to the plane equals the length of the projection of $\overrightarrow{BA}$ to $\vec{n}$

$\therefore$ L of $\perp$ from point $A(x_1, y_1, z_1)=\dfrac{|\overrightarrow{BA}\cdot\vec{n}|}{\|\vec{n}\|}$

$\boxed{\text{or}}\ =\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Example (1) find the length of the perpendicular drawn from point $(1,5,-4)$ to the plane whose equation $3x-y+2z=6$

Solution

$$L \text{ of the } \perp \text{ from } (\overset{x}{1},\overset{y}{5},\overset{z}{-4}) = \frac{|3x-y+2z-6|}{\sqrt{3^2+(-1)^2+2^2}} = \frac{|3\times1-5+2\times-4-6|}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

Example (2) find the length of the perpendicular drawn from the point $(1,-1,3)$ to the plane whose equation is $\vec{r}\cdot(2,2,-1)=5$

Solution

1st solution we find the Cartesian form of the equation of the plane $\vec{r}\cdot(2,2,-1)=5$

$$(x,y,z)\cdot(2,2,-1)=5$$

$$2x+2y-z-5=0$$

$$\therefore L \text{ of the } \perp \text{ from } (\overset{x}{1},\overset{y}{-1},\overset{z}{3}) = \frac{|2x+2y-z-5|}{\sqrt{(2)^2+(2)^2+(-1)^2}}$$

$$= \frac{|2\times1+2\times-1-3-5|}{3} = \frac{8}{3} \text{ L.unit.}$$

2nd solution we get point on the plane $\vec{r}\cdot(2,2,-1)=5$

let the point is $(0,0,z) \rightarrow \therefore (0,0,z)\cdot(2,2,-1)=5 \rightarrow z=-5$

$\therefore$ the point $B(0,0,-5)$ lies on the plane

$$\therefore \overrightarrow{BA} = \vec{A}-\vec{B} = (1,-1,3)-(0,0,-5) = (1,-1,8)$$

$$\therefore \text{the length of the } \perp = \frac{|\overrightarrow{BA}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|(1,-1,8)\cdot(2,2,-1)|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{8}{3}$$

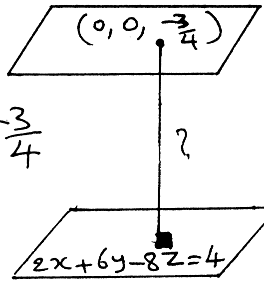
Example (3) Prove that the two planes $x+3y-4z=3$, $2x+6y-8z=4$ are parallel, then find the distance between them

Solution

To prove that the planes are parallel, we prove that their normal vectors are parallel. $\vec{n}_1 = (1, 3, -4)$, $\vec{n}_2 = (2, 6, -8)$

$\because \frac{a_1}{a_2} = \frac{1}{2}$, $\frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2}$, $\frac{c_1}{c_2} = \frac{-4}{-8} = \frac{1}{2}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ $\therefore$ the two planes are parallel



To find a point on the first plane, Put $x=0, y=0$

$x + 3y - 4z = 3 \rightarrow 0 + 3(0) - 4z = 3 \rightarrow z = \frac{-3}{4}$

$\therefore$ point $(0, 0, \frac{-3}{4}) \in$ the first plane.

L of the $\perp$ from $(0, 0, \frac{-3}{4}) = \frac{|2x + 6y - 8z - 4|}{\sqrt{2^2 + 6^2 + (-8)^2}}$

$= \frac{|2\times 0 + 6\times 0 - 8\times \frac{-3}{4} - 4|}{2\sqrt{26}} = \frac{\sqrt{26}}{26}$ unit length

Example (4) find the equation of the plane parallel to the plane $2x + y - 4z + 5 = 0$ which is $\sqrt{21}$ length unit distant from the point $(1, 2, 0)$

Solution

$\because$ the given plane is $2x + y - 4z + 5 = 0$

$\therefore$ the equation of the parallel plane is $2x + y - 4z + d = 0$

$\therefore$ L of $\perp$ from $(\overset{x}{1}, \overset{y}{2}, \overset{z}{0}) = \frac{|2x + y - 4z + d|}{\sqrt{(2)^2 + (1)^2 + (-4)^2}}$

$\frac{\sqrt{21}}{1} = \frac{|2\times 1 + 2 - 4\times 0 + d|}{\sqrt{21}}$

$\therefore |4 + d| = 21$

$\therefore 4 + d = \pm 21$

$4 + d = 21$, $4 + d = -21$

$d = 17$, $d = -25$

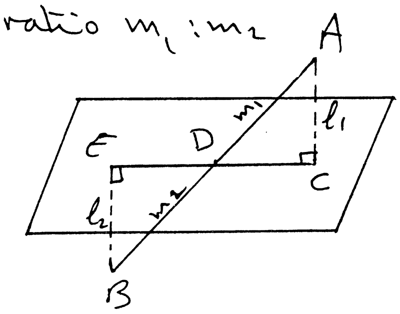
$\therefore$ there are two planes each is parallel to the given plane which are

$2x + y - 4z + 17 = 0$ and $2x + y - 4z - 25 = 0$

Example 5) find the ratio in which the plane $2x+3y-Z=5$ divides the distance between the two points $A(1,1,1)$, $B(3,2,-1)$ internally.

Solution

$\because$ the plane divides $\overline{AB}$ internally in the ratio $m_1 : m_2$



$\because \triangle ACD \sim \triangle BED$

$\therefore \frac{m_1}{m_2} = \frac{l_1}{l_2}$

$\because l_1 = \frac{|2(1)+3(1)-1-5|}{\sqrt{4+9+1}} = \frac{1}{\sqrt{14}}$

$l_2 = \frac{|2(3)+3(2)-(-1)-5|}{\sqrt{4+9+1}} = \frac{8}{\sqrt{14}}$

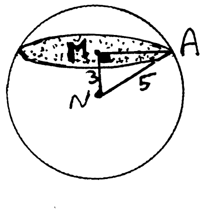
$\therefore l_1 : l_2 = 1 : 8$

$\therefore m_1 : m_2 = 1 : 8$

Example 6) find the area of the circle get from the intersecti[on] of the plane $2x-2y+Z=5$ and the sphere $(x-2)^2+(y-3)^2+(z+2)^2-25=0$

Solution

the center of the sphere is $N(2,3,-2)$ and it $r=5$



$\because \overline{MN} \perp$ the plane of the circle

$\therefore MN =$ L of the $\perp$ from $N(2,3,-2)$ to the plane

L of $\perp$ from $(\overset{x}{2},\overset{y}{3},\overset{z}{-2}) = \frac{|2x-2y+Z-5|}{\sqrt{2^2+(-2)^2+1^2}}$

$= \frac{|2\times2-2\times3+-2-5|}{\sqrt{9}} = 3$ unit length

from Phythagorus' theorem AM (r. of circle) $= \sqrt{25-9} = 4$ unit length

$\therefore$ area of the required circle $= \pi r^2 = 16\pi$ squared unit