Cubic roots of unity
$1, w, w^2$

If $z^3=1$ then the cubic roots in

① Algebraic form.

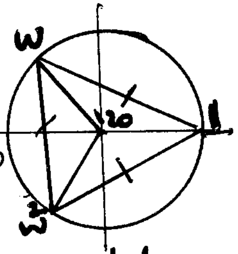
$1,\ \frac{-1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{-1}{2}-\frac{\sqrt{3}}{2}i$

② Trig form

$\cos 0 + i\sin 0 = 1$

$\cos 120 + i\sin 120 = \frac{-1}{2}+\frac{\sqrt{3}}{2}i = w$

$\cos -120 + i\sin -120 = \frac{-1}{2}-\frac{\sqrt{3}}{2}i = w^2$



the 3 roots are represented by vertecises of equilateral triangle whose side length $\sqrt{3}$

per $= 3\sqrt{3}$, area $= \frac{3\sqrt{3}}{4}$

$r=|z|=1$, $\theta = 120$ between each 2 consecutive roots

Properties of $1, w, w^2$

① $1+w+w^2=0$

$1+w=-w^2,\ 1+w^2=-w$

$w+w^2=-1$

② $1\times w\times w^2=1$

$w^3=1,\ \frac{1}{w^2}=w,\ \frac{1}{w}=w^2$

③ $w-w^2=\pm\sqrt{3}i$

$w^2-w=\pm\sqrt{3}i$

④ the conjugate of w is w^2

" " " w^2 is w

⑤ $w^{3n}=1,\ w^{3n+1}=w,\ w^{3n+2}=w^2$

⑥ If $z=w^n$ then $|z|=1$

⑦ $\frac{a+bw}{aw^2+b}=w$

⑧ the conjugate of w^2+1 is $w-1$

⑨ $\sum_{r=1}^{20}(-1)^{r+1}w^r = \ldots\ldots$

Solution

$(-1)^2w+(-1)^3w^2+\ldots\ldots$

$w-w^2+w^3-\ldots\ldots$

is geometric sequence

$a=w,\ r=-w,\ n=20$

$S_n=\frac{a(r^n-1)}{r-1}$

$=\frac{w((-w)^{20}-1)}{-w-1}$

$=\frac{w(w^{20}-1)}{w^2}=\frac{w^{21}-w}{w^2}$

$=w^{19}-\frac{1}{w}=w-w^2=\pm\sqrt{3}i$

⑩ $\sum_{r=1}^{100}w^r = w+w^2+\ldots\ldots w^{100}$

Geo. sequence

$a=w,\ r=w,\ n=100$

$S_{100}=\frac{w(w^{100}-1)}{w-1}=\frac{w\cancel{(w-1)}}{\cancel{w-1}}=w$

Notes:

① $i=\sqrt{-1},\ i^2=-1,\ i^3=-i,\ i^4=1 \rightarrow i^{4n}=1$

$i^{4n+1}=i,\ i^{8n+2}=i^2=-1$

② $i+i^2+i^3+i^4=0$

$i+i^2+i^3+i^4+\ldots\ldots i^{100}=0$

③ $(1+i)^2=2i,\ (1-i)^2=-2i$

$(1+i)^{2n}=(2i)^n$

④ If α,β are the roots of the quadratic eqn $ax^2+bx+c=0$ then

* sum of roots $=\frac{-b}{a}=\alpha+\beta$, Product of roots $=\alpha\beta=\frac{c}{a}$
* diff of roots $=\frac{\sqrt{b^2-4ac}}{a}$
* $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
* general form of quadratic equation $x^2-(\text{sum of roots})x+(\text{product of roots})=0$

⑤ If $z_1 = 1+\cos\theta + i\sin\theta$

$z_2 = 1-\cos\theta + i\sin\theta$

then $r_1 = 2\cos\frac{\theta}{2},\ \theta_1=\frac{\theta}{2}$

$r_2 = 2\sin\frac{\theta}{2},\ \theta_2 = (90-\frac{\theta}{2})$

⑥ If $z=e^{3+\frac{\pi}{6}i}$ then

$z=e^3 e^{\frac{\pi}{6}i}$

$r=e^3,\ \theta=\frac{\pi}{6}$