Finding the term containing x^k and the term Free of x in the expansion $(x+a)^n$

Example (1) In the expansion of $\left(\frac{x}{3} - \frac{3}{2x}\right)^8$

Find ① The term contains x^{-6}

② The term Free of x

Solution

$$T_{r+1} = {}^8C_r \left(\frac{-3}{2x}\right)^r \left(\frac{x}{3}\right)^{8-r}$$

$$= {}^8C_r (-1)^r (3)^r (2)^{-r} (x^{-r}) (x^{8-r}) (3^{-8+r})$$

$$= {}^8C_r (-1)^r (3^r) (2^{-r}) (3^{-8+r})\ x^{8-2r}$$

① Put $8-2r = -6 \rightarrow -2r = -6-8 \rightarrow -2r = -14 \rightarrow r = 7$

$\therefore T_8 = {}^8C_7 (-1)^7 (3^7)(2^{-7})(3^{-8+7})\ x^{-6}$

$\therefore T_8 = \frac{-729}{16} x^{-6}$

② To find the term Free of x put $8-2r=0 \rightarrow r=4$

$\therefore T_5 = {}^8C_4 (-1)^4 (3^4)(2^{-4})(3^{-8+4})\ x^0 = \frac{35}{8}$

Example (2) In the expansion $\left(2x^2 + \frac{1}{x}\right)^{12}$

Find ① The term contains x^9

② The term free of x

Solution

$$T_{r+1} = {}^{12}C_r \left(\frac{1}{x}\right)^r (2x^2)^{12-r}$$

$$= {}^{12}C_r (x^{-r})(2^{12-r})(x^{24-2r})$$

$$T_{r+1} = {}^{12}C_r (2^{12-r})(x^{24-3r})$$

① put $24-3r=9 \rightarrow -3r = 9-24 \rightarrow -3r = -15 \rightarrow r=5$

$\therefore T_6 = {}^{12}C_5 (2^{12-5})(x^{24-3\times5}) = 101376 x^9$

② To Find the term free of x Put $24-3r=0 \rightarrow r=8$

$\therefore T_9 = {}^{12}C_8 (2^{12-8})(x^{24-3\times8}) = 7920$

Example 3 In the expansion of $\left(2x - \frac{1}{2x^2}\right)^9$ find:

① the coefficient of x^3 ② The term free of x

③ Prove that the term does not contain x^2

Solution

$T_{r+1} = {}^9C_r \left(\frac{-1}{2x^2}\right)^r (2x)^{9-r} = {}^9C_r (-1)^r (2)^{-r} (2)^{9-r} x^{-2r} x^{9-r}$

$T_{r+1} = {}^9C_r (-1)^r (2)^{9-2r} x^{9-3r}$

① Put $9-3r=3 \rightarrow -3r = 3-9 \rightarrow -3r=-6 \rightarrow r=2$

$\therefore T_3 = {}^9C_2 (-1)^2 (2)^{9-4} x^{9-3\times2} = 1152x^3$

$\therefore$ the coefficient of x^3 is 1152

② Put $9-3r=0 \longrightarrow r=3$

$\therefore T_4 = {}^9C_3 (-1)^3 (2)^{9-2\times3} x^{9-3\times3} = -672$

③ Put $9-3r=2 \longrightarrow -3r = 2-9 \longrightarrow -3r=-7 \rightarrow r = \frac{7}{3} \notin N$

$\therefore$ this expansion does not contain x^2

Example 4 In the exponsion of $\left(ax + \frac{1}{bx}\right)^{10}$ according to the descending power of x, if the term Free of x equals the coefficient of the seventh term, Prove that $6ab=5$

Solution

$T_{r+1} = {}^{10}C_r \left(\frac{1}{bx}\right)^r (ax)^{10-r}$

$= {}^{10}C_r (b^{-r})(x^{-r})(a)^{10-r}(x^{10-r})$

$= {}^{10}C_r (b^{-r})(a)^{10-r}(x)^{10-2r}$

For the term free of $x \rightarrow 10-2r=0 \rightarrow r=5$

$\therefore$ Term free of $x = T_6 = {}^{10}C_5 (b)^{-5} (a)^{10-5} (x)^{10-2\times5} = 252a^5b^{-5}$

$T_7 = {}^{10}C_6 (b^{-6})(a^{10-6}) x^{10-2\times6} = {}^{10}C_6\, a^4 b^{-6} x^{-2}$

Coefficient of $T_7 = 210\, a^4 b^{-6}$

$\therefore 252\, a^5 b^{-5} = 210\, a^4 b^{-6} \rightarrow \div 42a^4b^{-6} \Longrightarrow \therefore 6ab=5$

Example 5 If n is positive integer, prove that there is no term free of x in the expansion of $(x^5 + \frac{1}{x^2})^n$ except if n is a multiple of 7, then find this term in the case of $n = 7$

Solution

$T_{r+1} = {}^nC_r \left(\frac{1}{x^2}\right)^r (x^5)^{n-r}$

$= {}^nC_r \, x^{-2r} \, x^{5n-5r}$

$= {}^nC_r \, x^{5n-7r}$

For the term free of x $\quad 5n - 7r = 0 \rightarrow 7r = 5n \rightarrow r = \frac{5n}{7}$

$\frac{5n}{7} \in Z^+$ if n is a multiple of 7

$\because r = \frac{5n}{7} \rightarrow$ put $n = 7 \rightarrow r = 5 \quad \therefore T_6 = {}^7C_5 = 21$

Example 6 In the expansion of $(x^2 + \frac{1}{x})^{3n}$ Find:

① The coefficient of the term which contains x^{3n}

② If $n = 6$, find the ratio between the coefficient of the term containing x^{3n} and the coefficient of the middle term

Solution

$T_{r+1} = {}^{3n}C_r \left(\frac{1}{x}\right)^r (x^2)^{3n-r}$

$T_{r+1} = {}^{3n}C_r (x^{-r})(x^{6n-2r})$

$T_{r+1} = {}^{3n}C_r \, x^{6n-3r}$

① put $6n - 3r = 3n \rightarrow \therefore 3n = 3r \quad \therefore r = n$

$\therefore$ the coefficient of the term containing x^{3n} is ${}^{3n}C_n$

② when $n = 6$ the coefficient of the term containing x^{3n}

$= {}^{3n}C_n = {}^{18}C_6$

the expansion $(x^2 + \frac{1}{x})^{18} \rightarrow T_{10}$ is the middle term $\quad (\frac{18}{2} + 1)$

$\therefore T_{10} = {}^{18}C_9 \left(\frac{1}{x}\right)^9 (x^2)^9 = {}^{18}C_9 (x^{-9})(x^{18}) = {}^{18}C_9 \, x^9$

$\therefore$ the ratio $= {}^{18}C_6 \div {}^{18}C_9 = \frac{21}{55}$

Example 7 In the expansion $\left(2+\frac{x}{3}\right)^9$, find the value of x which makes the two middle terms equal

Solution

The order of the two middle terms $\frac{9+1}{2}$, $\frac{9+3}{2}$ are T_5, T_6

$\because T_5 = T_6 \quad \therefore {}^9C_4 \left(\frac{x}{3}\right)^4 (2)^5 = {}^9C_5 \left(\frac{x}{3}\right)^5 (2)^4 \rightarrow \div \left(\frac{x}{3}\right)^4 (2)^4$

$2 = \frac{x}{3} \rightarrow x = 6$ $\boxed{\text{or}}$ Put $2 = \frac{x}{3}$

$\therefore x = 6$

Example 8 Find the coefficient of the middle term in the expansion $(1+3x+3x^2+x^3)^4$

Solution

$(1+3x+3x^2+x^3)^4 = \left[(1+x)^3\right]^4 = (1+x)^{12}$

the order of the middle term is $\frac{12}{2}+1 = 7$

$T_7 = {}^{12}C_6 x^6 \rightarrow \therefore$ Coefficient of $T_7 = {}^{12}C_6 = 924$

Example 9 Find the term free of x in the expansion

$$\frac{2}{x}\left(\frac{3}{x^2}+2x\right)^{10}$$

Solution

$T_{r+1} = \left(\frac{2}{x}\right) \times {}^{10}C_r (2x)^r \left(\frac{3}{x^2}\right)^{10-r}$

$T_{r+1} = {}^{10}C_r (2)^r (x)^r (3^{10-r}) (x^{-2})^{10-r} (2)(x^{-1})$

$= {}^{10}C_r (2)^{r+1} (3)^{10-r} (x)^{3r-21}$

put $3r-21=0 \rightarrow r=7$

$\therefore T_8 = {}^{10}C_7 (2)^8 (3)^3 = 829440$

Example 10 Find the value of term free of x in the expansion

$$(2+x)^5 + (1-x)^9$$

Solution

the value of the term free of x is $2^5 + 1^9 = \boxed{33}$

Example (11) In the expansion $\left(x^{K} + \frac{1}{x}\right)^{12}$ such that $K \in Z^{+}$

find ① the values of K for the expansion to have a term free of x

② the ratio between the term free of x and the cofficent of the middle term for the greatest value of K obtained

Solution

① let the Term free of x T_{r+1}

$T_{r+1} = {}^{12}C_r \left(\frac{1}{x}\right)^{r} \left(x^{K}\right)^{12-r} = {}^{12}C_r (x)^{-r} (x)^{12K-Kr}$

$\therefore T_{r+1} = {}^{12}C_r (x)^{12K-Kr-r}$

put $12K - Kr - r = 0$

$\therefore 12K - Kr = r$

$K(12-r) = r$

$\therefore K = \frac{r}{12-r}$, $K \in Z^{+}$ $\therefore K < 12$

$r = 11 \rightarrow K = \frac{11}{12-11} = 11$ $\therefore T_{12}$ free of x

$r = 10 \rightarrow K = \frac{10}{12-10} = 5$ $\therefore T_{11}$ " "

$r = 9 \rightarrow K = \frac{9}{12-9} = 3$ $\therefore T_{10}$ " "

$r = 8 \rightarrow K = \frac{8}{12-8} = 2$ $\therefore T_{9}$ " "

$r = 7 \rightarrow K = \frac{7}{12-7} = \frac{7}{5} \notin Z^{+}$

$r = 6 \rightarrow K = \frac{6}{12-6} = 1$ $\therefore T_{7}$ " "

② the greatest value of K obtained $= 11$ at $r = 11$

$\therefore$ the term free of x is $T_{12} = {}^{12}C_{11} = {}^{12}C_{1} = 12$

$\therefore$ the order of the middle terms $= T_7$ its coeff ${}^{12}C_6 = 924$

$\therefore \frac{\text{Term free of } x}{\text{coeff of middle term}} = \frac{12}{924} = \frac{1}{77}$

Example (12) Find ① the coefficient of x^2 in $(1-2x)^5(1+x)^{10}$
② the coeff of x^2 in the exp $(1+x-2x^2)^8$

Solution

① $(1-2x)^5(1+x)^{10}$

$= (1 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + \cdots\cdots)$

$\times (1 + {}^{10}C_1 x + {}^{10}C_2 x^2 + {}^{10}C_3 x^3 + \cdots\cdots)$

$\therefore$ the term contains x^2 is:

$1 \times {}^{10}C_2 x^2 + {}^5C_1(-2x) \times {}^{10}C_1 x + {}^5C_2(-2x)^2 \times 1$

$= 45x^2 + 5 \times -2x \times 10x + 10 \times 4x^2$

$= 45x^2 - 100x^2 + 40x^2 = -15x^2$

$\therefore$ the coefficient of x^2 is -15

<u>another solution</u>

$(1-2x)^5 \longrightarrow T_{r+1} = {}^5C_r(-2x)^r = {}^5C_r(-2)^r x^r \quad , r \le 5$

$(1+x)^{10} \longrightarrow T_{m+1} = {}^{10}C_m (x)^m \quad , m \le 10$

$\therefore T_{r+1} \times T_{m+1} = {}^5C_r(-2)^r(x)^r \times {}^{10}C_m (x)^m$

$= {}^5C_r \times {}^{10}C_m (-2)^r (x)^{r+m}$

put $r+m = 2$

$\therefore$ the coeff of $x^2 = {}^5C_r \times {}^{10}C_m (-2)^r$

r	0	1	2
m	2	1	0

$\therefore$ the coefficient of $x^2 = {}^5C_0 \times {}^{10}C_2 \times (-2)^0 + {}^5C_1 \times {}^{10}C_1 \times (-2)^1 +$
${}^5C_2 \times {}^{10}C_0 \times (-2)^2 = 45 + (-100) + 40 = -15$

② $[1 + x(1-2x)]^8$ its general term T_{r+1} , $r \le 8$

$T_{r+1} = {}^8C_r[x(1-2x)]^r = {}^8C_r(x)^r(1-2x)^r \longrightarrow ①$

$(1-2x)^r \longrightarrow$ its general term $T_{m+1} \longrightarrow m \le r$

$T_{m+1} = {}^rC_m(-2x)^m = {}^rC_m(-2)^m(x)^m$ by substituting in ①

$T_{r+1} = {}^8C_r(x)^r \times {}^rC_m(-2)^m(x^m) = {}^8C_r \times {}^rC_m(-2)^m(x)^{r+m}, m \le r \le 8$

Put $r+m = 2$

r	2	1	0
m	0	1	2

$\therefore$ the coeff of the term containing $x^2 = {}^8C_2\,{}^2C_0(-2)^0 + {}^8C_1\,{}^1C_1(-2)^1 = 12$

Ratio between consecutive terms in the binomial expansion $(x+a)^n$

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{a}{x}$$

Proof

$$T_{r+1} = {}^nC_r \, (a)^r \, (x)^{n-r}$$
$$T_r = {}^nC_{r-1} \, (a)^{r-1} \, (x)^{n-r+1}$$

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r (a)^r (x)^{n-r}}{{}^nC_{r-1} (a)^{r-1} (x)^{n-r+1}} = \frac{n-r+1}{r} \times \frac{a}{x}$$

Remark

$$\frac{\text{coefficient of } T_{r+1}}{\text{coefficient of } T_r} = \frac{n-r+1}{r} \times \frac{\text{coefficient of 2nd}}{\text{coefficient of 1st}}$$

Notice the difference between $\frac{{}^nC_r}{{}^nC_{r-1}}$ and $\frac{T_{r+1}}{T_r}$

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{n-(\text{the greater})+1}{\text{the greater}}$$

eg: $\frac{{}^7C_5}{{}^7C_4} = \frac{7-5+1}{5} = \frac{3}{5}$

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{\text{second}}{\text{first}} = \frac{n-(\text{the smaller})+1}{\text{the smaller}} \times \frac{\text{second}}{\text{first}}$$

eg: In the expansion of $(x+5)^7$

$$\frac{T_5}{T_4} = \frac{7-4+1}{4} \times \frac{5}{x} = \frac{5}{x}$$

Example ① In the expansion $(x+2y)^{12}$ find each of

① $\frac{T_3}{T_2}$ ② $\frac{T_8}{T_9}$ ③ $\frac{\text{coefficient of } T_7}{\text{coefficient of } T_8}$

④ $\frac{T_6}{T_4}$ ⑤ $\frac{\text{coefficient of } T_8}{\text{coefficient of } T_6}$

Solution

① $\frac{T_3}{T_2} = \frac{12-2+1}{2} \times \frac{2y}{x} = \frac{11}{2} \times \frac{2y}{x} = \frac{11y}{x}$

② $\frac{T_8}{T_9} = \frac{8}{12-8+1} \times \frac{x}{2y} = \frac{8}{5} \times \frac{x}{2y} = \frac{x}{y}$

③ $\frac{\text{coefficient of } T_7}{\text{coefficient of } T_8} = \frac{7}{12-7+1} \times \frac{1}{2} = \frac{7}{12}$

(4) $\frac{T_6}{T_4} = \frac{T_6}{T_5} \times \frac{T_5}{T_4} = \frac{12-5+1}{5} \times \frac{2y}{x} \times \frac{12-4+1}{4} \times \frac{2y}{x} = \frac{72y^2}{5x^2}$.

(5) $\frac{\text{coefficient of } T_8}{\text{coefficient of } T_6} = \frac{\text{Coefficient of } T_8}{\text{Coefficient of } T_7} \times \frac{\text{coefficient of } T_7}{\text{coefficient of } T_6}$

$$= \frac{12-7+1}{7} \times \frac{2}{1} \times \frac{12-6+1}{6} \times \frac{2}{1} = 4$$

Example 2 In the expansion $(x^2 + \frac{2}{x})^8$

First: Find the ratio between the fifth and the sixth terms. If this ratio equals 8 : 25, find the value of x

second: Prove that this expansion doesnot have a term free of x

Solution

$$\frac{T_6}{T_5} = \frac{8-5+1}{5} \times \frac{\frac{2}{x}}{x^2} = \frac{25}{8}$$

$$\frac{4}{5} \times \frac{2}{x^3} = \frac{25}{8}$$

$$\frac{2}{x^3} = \frac{25}{8} \div \frac{4}{5} \Rightarrow \frac{2}{x^3} = \frac{125}{32} \quad \therefore x = \frac{4}{5}$$

$$T_{r+1} = {}^8C_r \left(\frac{2}{x}\right)^r (x^2)^{8-r}$$

$$= {}^8C_r (2)^r (x)^{-r} (x)^{16-2r}$$

$$= {}^8C_r (2)^r (x)^{16-3r}$$

put $16 - 3r = 0 \rightarrow r = \frac{16}{3} \notin N$ $\quad \therefore$ there is no term free of x.

Example 3 In the expension of $(x+y)^8$ if $2T_5 = T_4 + T_6$ find $\frac{x}{y}$ numerically

Solution

$2T_5 = T_4 + T_6 \longrightarrow \div T_5$

$$\frac{T_4}{T_5} + \frac{T_6}{T_5} = 2$$

$$\frac{4}{8-4+1}\left(\frac{x}{y}\right) + \frac{8-5+1}{5}\left(\frac{y}{x}\right) = 2$$

$$\frac{4}{5} \times \frac{x}{y} + \frac{4}{5} \times \frac{y}{x} = 2 \longrightarrow \times 5xy$$

$$\frac{4}{\cancel{5}} \times \frac{x}{\cancel{y}} \times \cancel{5}x\cancel{y} + \frac{4}{\cancel{5}} \times \frac{y}{\cancel{x}} \times \cancel{5x}y = 10xy$$

$4x^2 + 4y^2 = 10xy \longrightarrow \div 2$

$2x^2 - 5xy + 2y^2 = 0$

$(2x - y)(x - 2y) = 0$

$2x - y = 0$, $2x = y$, $\frac{x}{y} = \frac{1}{2}$

$x - 2y = 0$, $x = 2y$, $\frac{x}{y} = 2$

Example (4) In the expansion of $(\sqrt{x}+\frac{1}{x})^8$, If $T_4, T_5, 25T_7$, and T_6 are proportional, find the value of x

Solution

$$\therefore \frac{T_4}{T_5} = \frac{25T_7}{T_6} \Rightarrow \frac{4}{8-4+1} \times \frac{\sqrt{x}}{\frac{1}{x}} = \frac{25(8-6+1)}{6} \times \frac{\frac{1}{x}}{\sqrt{x}}$$

$$\frac{4}{5}\sqrt{x} \times \frac{x}{1} = 25 \times \frac{1}{2} \times \frac{1}{x\sqrt{x}} \longrightarrow \times 10x\sqrt{x}$$

$$8x^2 \times x = 125 \rightarrow x^3 = \frac{125}{8} \rightarrow x = \frac{5}{2}$$

Example (5) If the coefficients of three consecutive terms of the expansion of $(1+x)^n$ are 35, 21, 7 according to the ascending power of x, find the value for each of n and the orders of these three terms.

Solution.

Let the terms are T_r, T_{r+1}, T_{r+2}

$$\frac{\text{coefficient of } T_{r+1}}{\text{coefficient of } T_r} = \frac{n-r+1}{r} = \frac{21}{35}$$

$$\therefore \frac{n-r+1}{r} = \frac{3}{5}$$

$$\therefore 5n - 5r + 5 = 3r$$

$$\therefore 5n - 8r = -5 \rightarrow (1)$$

$$\frac{\text{Coefficient of } T_{r+2}}{\text{coefficient of } T_{r+1}} = \frac{n-(r+1)+1}{r+1} = \frac{7}{21}$$

$$\therefore \frac{n-r}{r+1} = \frac{1}{3} \rightarrow 3n-3r = r+1$$

$$\therefore 3n - 4r = 1 \longrightarrow (2)$$

from (1) (2) $\therefore$ $\left.\begin{aligned} 5n-8r &= -5 \\ 3n-4r &= 1 \end{aligned}\right\}$ solve

$n = 7,\ r = 5$

Example (6) If the third, fourth and fifth terms of the expansion $(x+y)^n$ are 112, 448, 1120 respectively Find the value of each of n, y, x.

Solution

$$\because \frac{T_4}{T_3} = \frac{448}{112} \rightarrow \therefore \frac{n-3+1}{3} \times \frac{y}{x} = 4 \rightarrow \times 3$$

$$\therefore (n-2)\left(\frac{y}{x}\right) = 12 \rightarrow (1)$$

$$\frac{T_5}{T_4} = \frac{1120}{448} \longrightarrow \therefore \frac{n-4+1}{4} \times \frac{y}{x} = \frac{5}{2} \rightarrow \times 4$$

$$(n-3)\left(\frac{y}{x}\right) = 10 \rightarrow (2)$$

(1) ÷ (2)

$$\frac{n-2}{n-3} = \frac{6}{5}$$

$$6n - 18 = 5n - 10$$

$$n = 8$$

from ① $(8-2)\left(\frac{y}{x}\right)=12 \rightarrow \frac{y}{x}=2 \rightarrow \therefore y=2x$

$\because T_3 = {}^8c_2\,(y^2)(x)^6 = 112$

$\frac{8\times 7}{2\times 1}\,(4x^2)\,x^6 = 112$

$112x^8 = 112 \rightarrow x^{\overset{\text{even}}{8}} = 1 \rightarrow x = \pm 1 \rightarrow \therefore y = \pm 2$

Finding the greatest term

Example 7 Find the coefficient of the greatest term in the expansion

① $(1+x)^{10}$ ② $(1-x)^{17}$ ③ $(2x+3y)^{10}$ ④ $\left(1+\frac{1}{x}\right)^9$

Solution

① $(1+x)^{10 \rightarrow \text{even}} \rightarrow$ the greatest coeff $=$ Coeff. of the middle Term $T_6 = {}^{10}c_5 = 252$

② $(1-x)^{17 \rightarrow \text{odd}} \rightarrow$ the greatest coeff $=$ the coefficient of the two middle terms (T_9, T_{10})

$= {}^{17}c_8 = {}^{17}c_9$

$= 24310$

middle terms $= \frac{17+1}{2}, \frac{17+3}{2}$

T_9, T_{10}

coeff ${}^{17}c_8$ coef ${}^{17}c_9$

③ $(2x+3y)^{10}$

Put $\frac{T_{r+1}}{T_r} \geq 1 \rightarrow \therefore \frac{10-r+1}{r}\times\frac{3}{2} \geq 1 \rightarrow \therefore \frac{33-3r}{2r} \geq 1 \rightarrow \therefore 33-3r \geq 2r$

$\therefore 33 \geq 2r+3r \rightarrow \therefore 33 \geq 5r \rightarrow \therefore r \leq \frac{33}{5} = 6.6$

$\therefore r \leq 6.6 \notin Z \rightarrow \therefore r = 6$

$\therefore T_{6+1} = T_7$ has the greatest coefficient

$\therefore$ the greatest coefficient $= {}^{10}c_6 \times (3)^6 (2)^4 = 2449440$

④ $\left(1+\frac{1}{x}\right)^9$

Put $\frac{T_{r+1}}{T_r} \geq 1 \rightarrow \therefore \frac{9-r+1}{r}\times\frac{1}{1} \geq 1 \rightarrow \therefore \frac{10-r}{r} \geq 1 \rightarrow \therefore 10-r \geq r$

$\therefore 2r \leq 10 \rightarrow \therefore r \leq 5 \in Z \rightarrow r=5$ or $r=4$

$\therefore T_{4+1} = T_{5+1}$ and they are the greatest term (have the greatest coeff)

$\therefore T_5 = {}^9c_4\left(\frac{1}{x}\right)^4 (1)^5 = 126 \times \frac{1}{x^4} \rightarrow \therefore$ the greatest coeff $= 126$

Example [8] In the expansion $(2+\frac{x}{3})^9$

(1) find the value of x for which the two middle terms are equal

(2) find the coefficient of the greatest term

Solution

(1) The two middle terms are $T_{\frac{9+1}{2}}$, $T_{\frac{9+3}{2}} \rightarrow T_5, T_6$

$T_6 = T_5$ $\therefore$

$\therefore \frac{T_6}{T_5} = 1 \rightarrow \frac{9-5+1}{5} \times \frac{\frac{x}{3}}{2} = 1 \rightarrow \therefore \frac{x}{6} = 1 \rightarrow x = 6$

(2) Put $\frac{T_{r+1}}{T_r} \geq 1 \rightarrow \frac{9-r+1}{r} \times \frac{\frac{1}{3}}{2} \geq 1 \rightarrow \frac{10-r}{r} \times \frac{1}{6} \geq 1$

$\frac{10-r}{6r} \geq 1 \rightarrow 10-r \geq 6r \rightarrow 10 \geq 7r \rightarrow r \leq \frac{10}{7}$

$\therefore r \leq 1.4 \notin Z \rightarrow r = 1 \rightarrow T_{1+1} = T_2$ has the greatest coff.

$T_2 = {}^9C_1 \left(\frac{x}{3}\right)^1 (2)^8 = 9\,(x)(3^{-1})(2^8) = 768x$

$\therefore$ the coefficient of the greatest term is 768

Remarks

(1) the sum of the coefficients of the expansion of $(ax \pm by)^n$ is $(a \pm b)^n$ or put $x = 1, y = 1$ then calulate:

(2) To find the term free of x → find the general term then put the power of $x = 0$

or directely : the order of the term free of x

$$= \frac{\text{Power of binomial} \times \text{Power of the first term}}{\text{Power of first term} - \text{Power of 2nd term}} + 1$$

eg: the order of the term free of x in the binomial $\left(x^2 - \frac{1}{x^4}\right)^{12}$

is $\frac{12 \times 2}{2-(-4)} + 1 = 5$ $\therefore$ then T_5 is free of x

Ans. Ex. (1-2)

1) a) 14

2) d) 3

$(1+x)^5 = 1024$

$(1+x)^5 = (4)^5$

$1+x = 4$

$x = 3$

3) d) Zero

4) c) $16\ {}^{10}C_4$

$T_5 = {}^{10}C_4\ (1)^6\ (2x)^4 = 16\ {}^{10}C_4\ (x)^4$

$\therefore$ Coeff. $= 16\ {}^{10}C_4$

5) b) T_4

$24 - 4r = 12$

$24 - 12 = 4r$

$4r = 12$

$\therefore r = 3$

$\therefore T_{r+1} = T_4$

8) c) ${}^9C_5\ b^5$

$T_6 = {}^9C_5\ (bx)^5$

$= {}^9C_5\ (b)^5\ (x)^5$

$\therefore$ Coeff. $= {}^9C_5\ (b)^5$

9) a) $(a-b)^{12}$

$\because$ no. of terms equals =

$7+6 = 13$ terms

$\therefore$ power $= n = 13 - 1 = 12$

$\because$ terms are positive & negative $\therefore$ sign is negative

6) d) $a = 2b$

7) b) 2

The middle term $\frac{n+2}{2}$

$\frac{8n+2}{2} = 9$

$8n + 2 = 18$

$8n = 16$

$n = 2$

Second:

10) $(1+X)^8 = 256$

$(1+X)^8 = (2)^8$

$\therefore 1+X = 2$

$X = 1$

$(1+X)^8 = 256$

$(1+X)^8 = (-2)^8$

$1+X = -2$

$X = -3$

11) a) $(1.003)^5 = (1+0.003)^5 = 1 + {}^5C_1\,(0.003) + {}^5C_2\,(0.003)^2 + {}^5C_3\,(0.003)^3 + {}^5C_4\,(0.003)^4 + (0.003)^5 = 1.0151$

b) $(0.998)^7 = (1-0.002)^7 = 1 - {}^7C_1\,(0.002) + {}^7C_2\,(0.002)^2 - {}^7C_3\,(0.003)^3$

$+ {}^7C_4\,(0.002)^4 - - - - - - = 0.986$

c) $(1+0.01)^6 + (1-0.01)^6 = 2\,(T_1 + T_3 + T_5 + T_7)$

$= 2\,({}^6C_0\,(0.01) + {}^6C_2\,(0.01)^2 + {}^6C_4\,(0.01)^4 + {}^6C_6\,(0.01)^6)$

$\simeq 2.003$

d) $(1+0.02)^8 - (1-0.02)^8 = 2\,(T_2 + T_4 + T_6 + T_8)$

$= 2\,({}^8C_1\,(0.02) + {}^8C_3\,(0.02)^3 + {}^8C_5\,(0.02)^5 + {}^8C_7\,(0.02)^7)$

$\simeq 0.321$

12) $(1+\sqrt{3})^6 - (1-\sqrt{3})^6 = 480\sqrt{3}\;X$

$2\,(T_2 + T_4 + T_6) = 480\sqrt{3}\;X$

$2\,({}^6C_1\,(\sqrt{3}) + {}^6C_3\,(\sqrt{3})^3 + {}^6C_5\,(\sqrt{3})^5) = 480\sqrt{3}\,X$

$240\sqrt{3} = 480\sqrt{3}\;X \qquad \therefore X = 1/2$

14) a) $\left(\frac{2}{x}+\frac{x}{4}\right)^4$

$= \left(\frac{2}{x}\right)^4 + {}^4C_1\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right) + {}^4C_2\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^2 + {}^4C_3\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4$

$= \frac{16}{x^4} + \frac{16}{x^2} + 6 + x^2 + \frac{1}{16}x^4$

b) $\left(x-\frac{1}{x}\right)^5 = x^5 - {}^5C_1(x)^4\left(\frac{1}{x}\right) + {}^5C_2(x)^3\left(\frac{1}{x}\right)^2 - {}^5C_3(x)^2\left(\frac{1}{x}\right)^3$

$+ {}^5C_4(x)\left(\frac{1}{x}\right)^5 - \left(\frac{1}{x}\right)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$

c) $(x+\sqrt{2})^4 + (x-\sqrt{2})^4 = 2(T_1 + T_3 + T_5)$

$= 2\left({}^4C_0(x)^4 + {}^4C_2(x)^2(\sqrt{2})^2 + {}^4C_4(\sqrt{2})^4\right) = 2x^4 + 24x^2 + 8$

d) $(\sqrt{3}+2x)^5 - (\sqrt{3}-2x)^5 = 2(T_2 + T_4 + T_6)$

$= 2\left({}^5C_1(2x)(\sqrt{3})^4 + {}^5C_3(2x)^3(\sqrt{3})^2 + {}^5C_5(2x)^5\right)$

$= 2(90x + 240x^3 + 32x^5) = 180x + 480x^3 + 64x^5$

15) $T_5 = {}^nC_4\, x^4 = 1120$ $\qquad$ $T_3 = 28x^2$

${}^nC_2 x^2 = 28x^2$

${}^nC_2 = 28$

$\therefore n = 8$

$70x^4 = 1120$

$x^4 = 16$

$\therefore x = \pm 2$

16) $T_6 = {}^nC_5(x)^5$, $T_{10} = {}^nC_9(x)^9$

${}^nC_5 = {}^nC_9$

$n = 14$

17) $T_6 = {}^{10}C_5 (b)^5 (ax)^5 = {}^{10}C_5 (b)^5 (a)^5 (x)^5$

$\therefore {}^{10}C_5 (b)^5 (a)^5 = \frac{63}{8} \rightarrow 252(ab)^5 = \frac{63}{8}$

$(ab)^5 = \frac{1}{32} \rightarrow \therefore ab = \frac{1}{2} \qquad \therefore 2ab = 1$

18) middle term $= \frac{n}{2} + 1 = \frac{12}{2} + 1 = 7$

$T_7 = {}^{12}C_6 (2x^2)^6 (\frac{1}{2x})^6 = 924 x^6$

19) The middle terms are: $\frac{n+1}{2}$, $\frac{n+3}{2}$ $\qquad \therefore \frac{n+1}{2} = \frac{11+1}{2} = 6$

$\frac{n+3}{2} = \frac{11+3}{2} = 7$

$T_6 = {}^{11}C_5 (\frac{x^2}{2})^6 (\frac{-2}{x})^5 = -231 x^7$, $T_7 = {}^{11}C_6 (\frac{x^2}{2})^5 (\frac{-2}{x})^6 = 924 x^4$

20) T_4 from the end = T_7 from the beginning

$T_7 = {}^{9}C_6 (x)^3 (\frac{-1}{x})^6 = {}^{9}C_6 (x^3)(x^{-6}) = {}^{9}C_6 (x^{-3})$

$x^4 \cdot T_7 = {}^{9}C_6 (x^{-3})(x^4) = {}^{9}C_6 x = 84x$

21) The middle term is $= \frac{n}{2} + 1 = \frac{10}{2} + 1 = 6$

$T_6 = {}^{10}C_5 (x^2)^5 (\frac{1}{2x})^5 = {}^{10}C_5 (x)^{10} (\frac{1}{2})^5 (x)^{-5} = 252 \times \frac{1}{32} (x)^5$

$= \frac{63}{8} x^5$

$\therefore \frac{63}{8} x^5 = \frac{28}{7} \rightarrow x^5 = \frac{32}{243} \qquad \therefore x = \frac{2}{3}$

22) middle term is : $\frac{n}{2}+1 = \frac{10}{2}+1 = 6$

$$\frac{T_6}{T_5} = \frac{{}^{10}C_5 \left(\frac{2x}{3}\right)^5 \left(\frac{3}{2x}\right)^5}{{}^{10}C_4 \left(\frac{2x}{3}\right)^6 \left(\frac{3}{2x}\right)^4}$$

then put $x = 3$

$$\frac{T_6}{T_5} = \frac{{}^{10}C_5 (2)^5 (1/2)^5}{{}^{10}C_4 (2)^6 (1/2)^4} = \frac{1}{4} \times \frac{252}{210} = \frac{3}{10}$$

23) $\frac{T_5}{T_4} = \frac{-16}{15}$

$T_5 = {}^{15}C_4 (x)^{11} \left(\frac{1}{x}\right)^4 = {}^{15}C_4 (x^{11})(x^{-1})^4 = {}^{15}C_4 (x)^7$

$T_4 = {}^{14}C_3 (x)^{11} \left(\frac{-1}{x^2}\right)^3 = -{}^{14}C_3 (x)^{11} (x)^{-6} = -{}^{14}C_3\, x^5$

$$\frac{T_5}{T_4} = -\frac{{}^{15}C_4 (x)^7}{{}^{14}C_3 (x)^5} = -\frac{{}^{15}C_4}{{}^{14}C_3} x^2$$

$\because \frac{T_5}{T_4} = -\frac{16}{15}$

$\therefore \cancel{-} \frac{{}^{15}C_4}{{}^{14}C_3} x^2 = \cancel{-} \frac{16}{15}$ $\qquad \frac{1365}{364} x^2 = \frac{16}{15}$

$\therefore x^2 = \frac{64}{225}$ $\qquad \therefore x = \pm \frac{8}{15}$