Secondary 3

Algebra

Sheet 2

Binomial theorem

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Binomial theorem for positive integer power

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^4b + 6a^3b + 4a^2b + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$(a+b)^5 = a^5 + {}^5C_1a^4b + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + {}^5C_5b^5$

The expansion of a binomial

If a , $x \in R$, $n \in Z^+$ then

$(x+a)^5 = x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + \ldots\ldots\ldots\ldots\ldots + a^n$

$(x-a)^5 = x^n - {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 - \ldots\ldots\ldots\ldots\ldots + (-a)^n$

Remarks:

1) The number of terms of the expansion is (n+1) terms
2) The power x decrease and the power a increase
3) The sum of powers of x and powers of a in any term = n
4) The value of r in ${}^nC_r$ of each term is always decreased the order of the term by one

Example 1:

Find the expansion of $(x+2)^4$

SOLUTION

$(x+2)^4 = x^4 + {}^4C_1\, x^3(2)^1 + {}^4C_2\, x^2(2)^2 + {}^4C_3\, x\,(2)^3 + {}^4C_4\,(2)^4$

$= x^4 + 8\,x^3 + 24\,x^2 + 32\,x + 16$

Example 2:

Find the expansion of $(2x+3y)^4$

SOLUTION

$(2x+3y)^4 = (2x)^4 + {}^4C_1\,(2x)^3(3y)^1 + {}^4C_2\,(2x)^2(3y)^2 + {}^4C_3\,(2x)\,(3y)^3 + {}^4C_4\,(3y)^4$

$= 16x^4 + 96\,x^3y + 216\,x^2y^2 + 216\,xy^3 + 81y^4$

Example 3:

Find the expansion of $(x^2-1)^6$

SOLUTION

$$(x^2-1)^6 = (x^2)^6 - {}^6C_1\,(x^2)^5(1)^1 + {}^6C_2\,(x^2)^4(1)^2 - {}^6C_3\,(x^2)^3\,(1)^3 +$$
$$+{}^6C_4\,(x^2)^2(1)^4 - {}^6C_5\,(x^2)^1\,(1)^5 + {}^6C_6\,(1)^6$$
$$= x^{12} - 6\,x^{10} + 15\,x^8 - 20\,x^6 + 15\,x^4 - 6x^2 + 1$$

Special cases:

1) $(1+x)^n = 1 + {}^nC_1\,x + {}^nC_2\,x^2 + \ldots\ldots\ldots\ldots + x^n$
2) $(1-x)^n = 1 - {}^nC_1\,x + {}^nC_2\,x^2 - \ldots\ldots\ldots\ldots + (-x)^n$

Example 4:

Find the expansion of $(1+x)^6$, then use it to find the numerical value of the expansion ${}^6C_0 + {}^6C_1 + {}^6C_2 + \ldots\ldots\ldots\ldots + {}^6C_6$

SOLUTION

$$(1+x)^6 = 1 + {}^6C_1\,x + {}^6C_2\,x^2 + {}^6C_3\,x^3 + {}^6C_4\,x^4 + {}^6C_5\,x^5 + x^6$$

Put $x = 1$ in both sides

$$(1+1)^6 = 1 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + 1$$
$$(2)^6 = {}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$$

Example 5:

Find the expansion of $(1-x)^8$, then use it to find the numerical value of the expansion $1 - {}^8C_1 + {}^8C_2 - {}^8C_3 + \ldots\ldots\ldots\ldots + {}^8C_8$

SOLUTION

$$(1-x)^8 = 1 - {}^8C_1 x + {}^8C_2\,x^2 - {}^8C_3\,x^3 + {}^8C_4\,x^4 - {}^8C_5\,x^5 + {}^8C_6\,x^6$$
$$-{}^8C_7\,x^7 + {}^8C_8\,x^8$$

Put x = 1 in both sides

$$(1-1)^8 = 1 - {}^8C_1 + {}^8C_2 - {}^8C_3 + {}^8C_4 - {}^8C_5 + {}^8C_6 - {}^8C_7 + {}^8C_8 = 0$$

Example 6:

Find the value of $(1.01)^9$ by using the binomial theorem and approximate the result to three decimal numbers

SOLUTION

$$(1+0.01)^9 = 1 + {}^9C_1\left(\frac{1}{100}\right) + {}^9C_2\left(\frac{1}{100}\right)^2 + {}^9C_3\left(\frac{1}{100}\right)^3 + \dots\dots$$

$= 1 + 0.09 + 0.0036 + 0.000084 + \dots\dots$ terms less than 0.001

$= 1.0936 \approx 1.094$

Example 7:

Find the value of $(0.98)^{10}$ by using the binomial theorem and approximate the result to three decimal numbers

SOLUTION

$$(1-0.02)^{10} = 1 - {}^{10}C_1(0.02) + {}^{10}C_2(0.02)^2 - {}^{10}C_3(0.02)^3 + \dots\dots$$

$= 1 - (10 \times 0.02) + (45 \times 0.0004) - (120 \times 0.000008) + \dots$

$= 1 - 0.2 + 0.018 - 0.00096$

$= 0.81704 \approx 0.817$

Example 8:

Find the expansion of $(x-1)^5(x+1)^5$

SOLUTION

$$(x-1)^5(x+1)^5 = (x^2-1)^5$$

$= (x^2)^5 - {}^5C_1(x^2)^4 + {}^5C_2(x^2)^3 - {}^5C_3(x^2)^2 + {}^5C_4\,x^2$

$= x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1$

Example 9:

Find the expansion of $\left(1 + x - \frac{1}{x}\right)^3$

SOLUTION

$$\left((1 + x) - \frac{1}{x}\right)^3 = (1 + x)^3 - {}^3C_1\,(1 + x)^2\left(\frac{1}{x}\right) + {}^3C_2\,(1 + x)\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^3$$

$$= (1 + {}^3C_1\,x + {}^3C_2\,x^2 + x^3) - 3(1 + 2x + x^2)\left(\frac{1}{x}\right) + 3(1 + x)\left(\frac{1}{x^2}\right) - \left(\frac{1}{x^3}\right)$$

$$= 1 + 3x + 3x^2 + x^3 - \frac{3}{x} - 6 - 3x + \frac{3}{x^2} + \frac{3}{x} - \frac{1}{x^3}$$

$$= x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$$

Multiple choice questions:

1) If $1 + 7x + \frac{7\times6}{2\times1}\,x^2 + \ldots\ldots\ldots + x^7 = 2187$, then $x = \ldots\ldots\ldots\ldots$

(a) 4 (b) 2 (c) 8 (d) 10

SOLUTION

$$1 + 7x + {}^7C_2\;x^2 + \ldots\ldots\ldots + x^7 = (1 + x)^7 = 2187 = 3^7$$

$$\therefore 1 + x = 3 \quad \rightarrow \quad x = 2$$

2) If $(1 - x)^8 + 16\,x\,(1 - x)^7 + 112\,x^2\,(1 - x)^6 + \ldots\ldots\ldots + 256\,x^8 = 0$,
then $x = \ldots\ldots\ldots\ldots$

(a) 2 (b) −2 (c) −1 (d) 1

SOLUTION

$$(1 - x)^8 + 16\,x\,(1 - x)^7 + 112\,x^2\,(1 - x)^6 + \ldots\ldots\ldots + (2x)^8 = 0$$

$$\therefore [(1 - x) + 2x]^8 = [1 + x]^8 = 0 \quad \rightarrow \quad 1 + x = 0 \quad \rightarrow \quad x = -1$$

The general term of the expansion of a binomial

The general term of the expansion of a binomial $(x+a)^n$ is denoted by

T_{r+1} such that $r \in N$, $r \leq n$

$$T_{r+1} = {}^nC_r \, (\text{second})^r \, (\text{first})^{n-r}$$

$$T_{r+1} = {}^nC_r \quad a^r \quad x^{n-r}$$

Notice 1 :

In the expansion $(1+x)^n$

$T_{r+1} = {}^nC_r \, x^r$, $T_r = {}^nC_{r-1} \, x^{r-1}$

Coefficient of x^r is ${}^nC_r$ but coefficient of T_r is ${}^nC_{r-1}$

Notice 2 :

$$(x+a)^n + (x-a)^n = 2\,(T_1 + T_3 + T_5 + \dots\dots\dots)$$

$$(x+a)^n - (x-a)^n = 2\,(T_2 + T_4 + T_6 + \dots\dots\dots)$$

Multiple choice questions:

1) In the expansion $\left(x + \frac{2}{x}\right)^8$ according to the descending power of x , the coefficient of the sixth term = ...

(a) 940 (b) 2179 (c) 1972 (d) 1792

SOLUTION

$$T_6 = {}^8C_5 \left(\frac{2}{x}\right)^5 (x)^3 = {}^8C_5 \, (2)^5 (x)^{-5} \, (x)^3 = {}^8C_5 \, (32) \, (x)^{-2}$$

coefficient of $T_6 = {}^8C_5 \, (32) = 1792$

2) In the expansion $\left(3x^2 - \frac{1}{2x}\right)^{13}$ according to the descending power of x, then the tenth term from the end = ...

(a) $\frac{143\times3^9}{8}x^{14}$ (b) $\frac{715\times3^9}{16}x^{14}$ (c) $\frac{715\times3^8}{16}x^{14}$ (d) $\frac{715\times3^9}{16}x^{13}$

SOLUTION

the tenth term from the end in the expansion $\left(3x^2 - \frac{1}{2x}\right)^{13}$

is the term from the beginning of the expansion $\left(-\frac{1}{2x} + 3x^2\right)^{13}$

$$T_{10} = {}^{13}C_9\,(3x^2)^9\left(-\frac{1}{2x}\right)^4 = {}^{13}C_4\,(3)^9\,x^{18}\,(-1)^4(2)^{-4}(x)^{-4}$$

$$= \frac{715\times3^9}{16}x^{14}$$

ANOTHER SOLUTION

T_{10} from the end = $T_{14-10+1} = T_5$ from the beginning of the expansion $\left(3x^2 - \frac{1}{2x}\right)^{13}$

$$T_5 = {}^{13}C_4\left(-\frac{1}{2x}\right)^4(3x^2)^9 = {}^{13}C_4\left(-\frac{1}{2}\right)^4 x^{-4}\,(3)^9\,x^{18} = \frac{715\times3^9}{16}x^{14}$$

3) In the expansion $\left(2x - \frac{-1}{3x^2}\right)^{11}$ according to the descending power of x, then the fourth term from the end = ...

(a) $\frac{400}{2187}x^{13}$ (b) $\frac{440}{2180}x^{-13}$ (c) $\frac{440}{2187}x^{13}$ (d) $\frac{440}{2187}x^{-13}$

SOLUTION

T_4 from the end = $T_{12-4+1} = T_9$ from the beginning

$$T_9 = {}^{11}C_8\left(\frac{-1}{3x^2}\right)^8(2x)^3 = {}^{11}C_3\left(-\frac{1}{3}\right)^8 x^{-16}\,(2)^3\,x^3 = \frac{440}{2187}x^{-13}$$

4) In the expansion $(1+x)^n$ if coefficient of $T_4 = 120$ then $n = \dots\dots\dots$

(a) 10 (b) 21 (c) 9 (d) 11

SOLUTION

$T_4 = {}^{n}C_3\,(x)^3$

coefficient of $T_4 = {}^{n}C_3 = 120$

$\therefore n = 10$

5) In the expansion

$(3+x)^{11} - {}^{11}C_1(3+x)^{10}(1-2x) + {}^{11}C_2(3+x)^9(1-2x)^2 - \dots\dots\dots - (1-2x)^{11}$

the fifth term $= \dots\dots\dots\dots\dots$

(a) $3421440\,x^4$ (b) $3421440\,x^7$ (c) $31440\,x^4$ (d) $3144\,x^7$

SOLUTION

The expansion represents the expansion of $[(3+x)-(1-2x)]^{11}$

$= [2+3x]^{11}$

$T_5 = {}^{11}C_4(3x)^4(2)^7 = 330 \times (3)^4 \times x^4 \times (2)^7 = 3421440\,x^4$

6) In the expansion

$(1-x)^8 + 24\,x\,(1-x)^7 + 252\,x^2\,(1-x)^6 + \dots\dots\dots + 6561\,x^8$,

then T_6 at $x = 1$ is $\dots\dots\dots$

(a) 940 (b) 2179 (c) 1972 (d) 1792

SOLUTION

$(1-x)^8 + 24\,x\,(1-x)^7 + 252\,x^2\,(1-x)^6 + \dots\dots\dots + (3x)^8$

$= [(1-x)+3x]^8 = [1+2x]^8$

$T_6 = {}^{8}C_5(2x)^5 = {}^{8}C_5\big(2(1)\big)^5 = 1792$

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7) In the expansion $(3+x)^n$ according to ascending power of x, if the ratio between the sixth term and eighth term is 9 : 4 , then n = ……… when $x = 2$

(a) 9 (b) 21 (c) 12 (d) 11

SOLUTION

$$T_6 = {}^nC_5(x)^5(3)^{n-5} \qquad , \qquad T_8 = {}^nC_7(x)^7(3)^{n-7}$$

$$\frac{T_8}{T_6} = \frac{{}^nC_7(x)^7(3)^{n-7}}{{}^nC_5(x)^5(3)^{n-5}} = \frac{4}{9} = \frac{{}^nC_7}{{}^nC_6} \times \frac{{}^nC_6}{{}^nC_5} \; x^2 \times 3^{n-7-n+6}$$

ReMEMbeR: $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

$$= \frac{n-7+1}{7} \times \frac{n-6+1}{6} \; x^2 \times 3^{-2} = \frac{4}{9}$$

$$= \frac{(n-6)(n-5)}{42} \; \frac{x^2}{9} = \frac{4}{9} \qquad \text{at x = 2} \rightarrow \frac{(n-6)(n-5)}{42} \; \frac{4}{9} = \frac{4}{9}$$

$$\frac{(n-6)(n-5)}{42} = 1 \rightarrow (n-6)(n-5) = 42$$

$$6 \times 7 = 42 \rightarrow n-6 = 6 \rightarrow n = 12$$

8) In the expansion of $(1+x-x^2)^9$, the coefficient of x^{10} = …………

(a) 1257 (b) 210 (c) 135 (d) 117

SOLUTION

In the expansion of $\left(1+x(1-x)\right)^9$

$$T_{r+1} = {}^9C_r\left(x(1-x)\right)^r = {}^9C_r\, x^r(1-x)^r \qquad , r \le 9$$

In the expansion of $(1-x)^r \rightarrow T_{m+1} = {}^rC_m(-x)^m \qquad , m \le r$

$$T_{r+1} = {}^9C_r \; x^r \; {}^rC_m(-x)^m = (-1)^m \times {}^9C_r \times {}^rC_m (x)^{r+m}$$

To find the coefficient of x^{10} put r + m = 10 where $m \le r \le 9$

r = 9	r = 8	r = 7	r = 6	r = 5
m = 1	m = 2	m = 3	m = 4	m = 5

coefficient of x^{10} =

$$-{}^9C_9 \times {}^9C_1 + {}^9C_8 \times {}^8C_2 - {}^9C_7 \times {}^7C_3 + {}^9C_6 \times {}^6C_4 - {}^9C_5 \times {}^5C_5 = 117$$

9) In the expansion of $(1+x+x^2)^5$, the coefficient of $x^4 =$ ....

(a) 125 (b) 45 (c) 135 (d) 54

SOLUTION

In the expansion of $\left(1+x(1+x)\right)^5$

$T_{r+1} = {}^5C_r\left(x\,(1+x)\right)^r = {}^5C_r\, x^r(1+x)^r = \quad , r \le 5$

In the expansion of $(1+x)^r \rightarrow T_{m+1} = {}^rC_m(x)^m \quad , m \le r$

$T_{r+1} = {}^5C_r\ \ x^r\ \ {}^rC_m(x)^m = {}^5C_r \times {}^rC_m(x)^{r+m}$

To find the coefficient of x^4 put r + m = 4 where m ≤ r ≤ 5

r = 4	r = 3	r = 2
m = 0	m = 1	m = 2

coefficient of $x^4 = {}^5C_4 \times {}^4C_0 + {}^5C_3 \times {}^3C_1 + {}^5C_2 \times {}^2C_2 = 45$

Example 1:

In the expansion $\left(2x^2+\frac{1}{2}\right)^7$ according to the descending power of x, Find each of T_3, T_6 and if $T_3 = T_6$, Find the value of x

SOLUTION

$T_3 = {}^7C_2\left(\frac{1}{2}\right)^2(2x^2)^5$

$= 168\,x^{10}$

$T_6 = {}^7C_5\left(\frac{1}{2}\right)^5(2x^2)^2$

$= \frac{21}{8}x^4$

$T_3 = T_6$

$168\,x^{10} = \frac{21}{8}x^4$

$\frac{x^{10}}{x^4} = \frac{21}{8} \div 168$

$x^6 = \frac{1}{64} \rightarrow x = \pm\frac{1}{2}$

Example 2:

In the expansion $(1+x)^n$ if $T_2 = \frac{-10}{3}$, $T_3 = \frac{40}{9}$, find n, x then find T_4

SOLUTION

$T_2 = {}^nC_1\, x = \frac{-10}{3} \quad \rightarrow \quad n\,x = \frac{-10}{3} \qquad \rightarrow 1$

$T_3 = {}^nC_2\, x^2 = \frac{40}{9}$

$\frac{n(n-1)}{2\times 1} x^2 = \frac{40}{9} \rightarrow \times 2 \quad \rightarrow \quad n(n-1)\, x^2 = \frac{80}{9} \quad \rightarrow 2$

By squaring 1 $\rightarrow n^2 x^2 = \frac{100}{9} \qquad \rightarrow 3$

$2 \div 3 \rightarrow \frac{n(n-1)x^2}{n^2 x^2} = \frac{(n-1)}{n} = \frac{4}{5} \quad \rightarrow \quad n = 5$

$5\,x = \frac{-10}{3} \quad \rightarrow \quad x = \frac{-2}{3}$

$T_4 = {}^nC_3\, x^3 = {}^5C_3 \left(\frac{-2}{3}\right)^3 = \frac{-80}{27}$

Example 3:

Find in the simplest from $(x+2)^6 + (x-2)^6$

SOLUTION

$(x+2)^6 + (x-2)^6 = 2\,(T_1 + T_3 + T_5 + T_7)$

$= 2\,(x^6 + {}^6C_2\,(2)^2\,x^4 + {}^6C_4\,(2)^4\,x^2 + {}^6C_6\,(2)^6)$

$= 2\,(x^6 + 60\,x^4 + 240\,x^2 + 64)$

Example 4:

Find in the simplest from $\left(1+\sqrt{x}\right)^5-\left(1-\sqrt{x}\right)^5$

SOLUTION

$$\left(1+\sqrt{x}\right)^5-\left(1-\sqrt{x}\right)^5 = 2\,(T_2 + T_4 + T_6)$$

$$= 2\,({}^5C_1\,(\sqrt{x}) + {}^5C_3\,(\sqrt{x})^3 + {}^5C_5\,(\sqrt{x})^5)$$

$$= 2\left[5\sqrt{x} + 10(\sqrt{x})^3 + (\sqrt{x})^5\right]$$

$$= 10\sqrt{x} + 20(\sqrt{x})^3 + 2(\sqrt{x})^5$$

Example 5:

Find to the nearest three decimals $(1.03)^8 + (0.97)^8$ using the binomial theorem

SOLUTION

$$(1+0.03)^8 + (1-0.03)^8 = 2\,(T_1 + T_3 + T_5 + T_7 + T_9)$$

$$= 2\,(1 + {}^8C_2\,(0.03)^2 + {}^8C_4\,(0.03)^4 + {}^8C_6\,(0.03)^6 + {}^8C_8\,(0.03)^8)$$

$$= 2\,(1 + 0.0252 + 0.0000567 + \dots\dots\dots)$$

$$= 2 + 0.0504 + 0.0001134 + \dots\dots\dots$$

$$= 2.0505134 \cong 2.051$$

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Example 6:

If $(1+c\,x)^n = 1 + 20\,x + a_1x^2 + a_2x^3 + \dots\dots + a_{n-1}x^n$ and $16\,a_1 = 3\,a_2$, find the value of n, c where $c \neq 0$

SOLUTION

$$(1+c\,x)^n = 1 + {}^nC_1(cx) + {}^nC_2(cx)^2 + {}^nC_3(cx)^3 + \dots\dots + (cx)^n$$

$$= 1 + {}^nC_1c\,x + {}^nC_2c^2\,x^2 + {}^nC_3c^3\,x^3 + \dots\dots + c^nx^n$$

$$= 1 + 20\,x + a_1\,x^2 + a_2\,x^3 + \dots\dots + a_{n-1}x^n$$

$${}^nC_1 \times c = 20 \rightarrow n\,c = 20 \rightarrow c = \frac{20}{n}$$

$${}^nC_2\,c^2 = a_1 \quad , \quad {}^nC_3\,c^3 = a_2$$

$$16\,a_1 = 3\,a_2$$

$$16\,{}^nC_2\,c^2 = 3\,{}^nC_3\,c^3 \rightarrow \div c^2$$

$$16\,{}^nC_2 = 3\,{}^nC_3\,c$$

$$16\,{}^nC_2 = 3\,{}^nC_3 \times \frac{20}{n} \rightarrow \div {}^nC_2$$

$$16 = 3 \times \frac{{}^nC_3}{{}^nC_2} \times \frac{20}{n}$$

$$16 = 3 \times \frac{n-3+1}{3} \times \frac{20}{n}$$

$$16 = \frac{20\,(n-2)}{n} \rightarrow \times n$$

$$16n = 20\,(n-2) \rightarrow n = 10$$

$$\therefore c = 2$$

ANOTHER SOLUTION

$${}^nC_1 \times c = 20$$

$$n\,c = 20 \rightarrow (1)$$

$${}^nC_2\,c^2 = a_1$$

$$\frac{n(n-1)}{2\times1}c^2 = a_1 \rightarrow \times 2$$

$$n(n-1)c^2 = 2\,a_1 \rightarrow (2)$$

$${}^nC_3\,c^3 = a_2$$

$$\frac{n(n-1)(n-2)}{3\times2\times1}c^3 = a_2 \rightarrow \times 6$$

$$n(n-1)(n-2)c^3 = 6\,a_2 \rightarrow (3)$$

By squaring 1 $\rightarrow n^2c^2 = 400 \rightarrow (4)$

$$4 \div 2 \rightarrow \frac{n^2c^2}{n\,(n-1)\,c^2} = \frac{n}{n-1} = \frac{400}{2\,a_1} \rightarrow \text{I}$$

By cubing 1 $\rightarrow n^3c^3 = 8000 \rightarrow (5)$

$$3 \div 5 \rightarrow \frac{n\,(n-1)(n-2)c^3}{n^3c^3} = \frac{(n-1)(n-2)}{n^2} = \frac{6\,a_2}{8000} \rightarrow \text{II}$$

By (I×II)

$$\frac{n}{n-1} \times \frac{(n-1)(n-2)}{n^2} = \frac{400}{2\,a_1} \times \frac{6\,a_2}{8000}$$

$$\frac{(n-2)}{n} = \frac{400}{a_1} \times \frac{3\,a_2}{8000}$$

$16\,a_1 = 3\,a_2$

$$\frac{(n-2)}{n} = \frac{400}{a_1} \times \frac{16\,a_1}{8000} = \frac{4}{5}$$

$5n - 10 = 4n \quad \rightarrow \quad n = 10 \quad \rightarrow \quad c = 2$

Example 7:

In the expansion of $(1 + k\,x)^{10}$ if the coefficient of the third term is 180 and the fifth term is 210, Find the value of k and x where k is positive integer

SOLUTION

coefficient of the $T_3 = {}^{10}C_2(k)^2 = 180$

$k = 2$ or $k = -2$ **(refused)**

$T_5 = {}^{10}C_4(kx)^4 = 210$

$= 210\,(kx)^4 = 210$

$= (\pm 2\,x)^4 = 1$

$\pm 2\,x = 1 \quad \rightarrow \quad x = \pm \frac{1}{2}$

The middle term of the expansion $(x+a)^n$

In the expansion $(x+a)^n$, number of terms of the expansion = n + 1

1) If n is even, then the number of terms of the expansion is odd and the expansion has only one middle term of order $\frac{n+2}{2}$ or $\frac{n}{2}+1$

2) If n is odd, then the number of terms of the expansion is even and the expansion has two middle terms of order $\frac{n+1}{2}$, $\frac{n+3}{2}$

Multiple choice questions:

1) In the expansion $\left(2x+\frac{1}{2x^2}\right)^{12}$ the middle term is ....

(a) $924\,x^{-6}$ (b) $924\,x^{-12}$ (c) $492\,x^{-6}$ (d) $924\,x^{6}$

SOLUTION

The power is even ∴ the order of the middle term $=\frac{12}{2}+1=7$

$$T_7={}^{12}C_6\left(\frac{1}{2x^2}\right)^6(2x)^6={}^{12}C_6\left(\frac{1}{2}\right)^6(x)^{-12}(2)^6(x)^6$$
$$={}^{12}C_6\,(x)^{-6}=924\,x^{-6}$$

2) Find the middle term of the expansion $\left(x^2+\frac{1}{2x}\right)^{10}$ if the value of this term is $\frac{28}{27}$, then the value $x=$ ....

(a) $\frac{2}{4}$ (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

the order of the middle term $=\frac{10}{2}+1=6$

$$T_6={}^{10}C_5\left(\frac{1}{2x}\right)^5(x^2)^5={}^{10}C_5\,(2)^{-5}(x)^{-5}\;x^{10}=\frac{63}{8}\;x^5$$

$$\frac{63}{8}\;x^5=\frac{28}{27}\;\rightarrow\;x^5=\frac{32}{243}\;\rightarrow x=\frac{2}{3}$$

3) In the expansion $\left(\frac{x^2}{3}+\frac{3}{x}\right)^{15}$ the middle term equal ....

(a) $2145\ x^9, 19305\ x^6$

(b) $19305\ x^6\ , 135135\ x^3$

(c) $135135\ x^3, 729729$

(d) $\frac{5005\ x^{12}}{27}, 2145\ x^9$

SOLUTION

the order of the middle term $=\frac{15+1}{2}=8$ and $\frac{15+3}{2}=9$

$$T_8={}^{15}C_7\left(\frac{3}{x}\right)^7\left(\frac{x^2}{3}\right)^8={}^{15}C_7\ 3^7\ (x)^{-7}\ x^{16}\ 3^{-8}={}^{15}C_7\ \frac{1}{3}\ x^9=2145\ x^9$$

$$T_9={}^{15}C_8\left(\frac{3}{x}\right)^8\left(\frac{x^2}{3}\right)^7={}^{15}C_7\ 3^8\ (x)^{-8}\ x^{14}\ 3^{-7}={}^{15}C_8\ 3\ x^6=19305\ x^6$$

4) If the middle term of the expansion $\left(3x^2+\frac{2}{3x}\right)^8$ equal 17920, then the value of $x=$ ....

(a) 2

(b) 4

(c) 2 , – 2

(d) 4 , – 4

SOLUTION

the order of the middle term $=\frac{8}{2}+1=5$

$$T_5={}^{8}C_4\left(\frac{2}{3x}\right)^4(3x^2)^4={}^{8}C_4(2)^4(3)^{-4}(x)^{-4}(3)^4(x)^8=1120\ x^4$$

$1120\ x^4=17920$

$x^4=16 \quad\rightarrow\quad x=\pm 2$

5) In the expansion: $(3+2x)^8+(3-2x)^8$ Then the middle term equal ....

(a) $32256\,x^6$ (b) $181440\,x^4$ (c) $1810\,x^4$ (d) $163296\,x^2$

SOLUTION

the order of the middle term $=\frac{8}{2}+1=5$

$T_5={}^8C_4\,(2x)^4\,(3)^4+{}^8C_4(-2x)^4(3)^4={}^8C_4\,(2x)^4\,(3)^4+{}^8C_4(2x)^4(3)^4$

$=2\times{}^8C_4\,(2)^4\,(x)^4\,(3)^4=181440\,x^4$

6) In the expansion: $\left(2\sqrt{x}+\frac{1}{2\sqrt{x}}\right)^{10}+\left(2\sqrt{x}-\frac{1}{2\sqrt{x}}\right)^{10}$

Then the middle term equal ....

(a) $1680\,x$ (b) $\frac{108}{x}$ (c) $5760\,x^3$ (d) zero

SOLUTION

the order of the middle term $=\frac{10}{2}+1=6$

$T_6={}^{10}C_5\left(\frac{1}{2\sqrt{x}}\right)^5\left(2\sqrt{x}\right)^5+{}^{10}C_5\left(-\frac{1}{2\sqrt{x}}\right)^5\left(2\sqrt{x}\right)^5$

$={}^{10}C_5\left(\frac{1}{2\sqrt{x}}\right)^5\left(2\sqrt{x}\right)^5-{}^{10}C_5\left(\frac{1}{2\sqrt{x}}\right)^5\left(2\sqrt{x}\right)^5=0$

7) In the expansion: $(1+x)^6+6x(1+x)^5+15\,x^2(1+x)^4+\ldots\ldots\ldots\ldots+x^6$

Then the middle term equal ....

(a) $240\,x^4$ (b) $160\,x^3$ (c) $576\,x^3$ (d) $60\,x^2$

SOLUTION

The expansion $[(1+x)+x]^6=[1+2x]^6$

the order of the middle term $=\frac{6}{2}+1=4$

$T_4={}^6C_3\,(2x)^3={}^6C_3\,(2)^3\,(x)^3=160\,x^3$

Example(8):

If the two middle terms of the expansion $(3x + 2y)^{13}$ are equal,

Prove that $\frac{x}{y} = \frac{2}{3}$

SOLUTION

the order of the middle term $= \frac{13+1}{2} = 7$ and $\frac{13+3}{2} = 8$

$$T_8 = T_7$$

$$^{13}C_7\,(2y)^7(3x)^6 = {}^{13}C_6\,(2y)^6(3x)^7$$

$$\frac{(2y)^7}{(2y)^6} = \frac{(3x)^7}{(3x)^6}$$

$$2y = 3x$$

$$\frac{x}{y} = \frac{2}{3}$$

Or directly for choose problems:

If the two middle terms are equal

$\therefore 3x = 2y$

$\frac{x}{y} = \frac{2}{3}$